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An Analysis of the Detection of Repeated Signals in Noise by Binary Integration*

J. V. HARRINGTON†

Summary—**An analysis of the detection of repetitive signals in noise by binary integration techniques is made. An expression for the effective signal-to-noise ratio of the quantized video is obtained and is shown to apply to any half-wave second detector. A comparison of analog and digital integration is made, and it is further shown that digital integration is, at most, 1.9 db poorer due to the quantization loss. However, the loss due to nonideal analog integration can make the two types equivalent. The optimum settings for quantizer and counter thresholds are derived, and expressions for the final-detection and false-alarm probabilities are determined. Lastly, the results are modified to include the effect of nonuniform amplitudes in the set of signals being quantized and integrated.**

Introduction

RECENTLY, signal-integration techniques have come into use as a means of improving the detection of repetitive signals in noise.[1] However, the methods used to date for obtaining the desired signal storage, which is a necessary part of the integration process, have been primarily analog in nature. For example, delay-line integrators and storage-tube integrators have been developed which integrate by remembering the waveform of the signal in noise and then superimposing successive samples to obtain the desired improvement. The theory is that, when successive repetition intervals of any radar video are added, the noise voltage increases roughly as the square root of the number of additions, and the signal increases as the number of additions. Hence the relative signal-to-noise ratio should increase as the square root of the number of samples added. This has been borne out by experiment and has been well reported in the literature. There are, however, several disadvantages to analog integrators: the rather large number of memory elements required to store the waveform of the signal, and the relatively short memory times; thus, in the radar case, an addition of some ten or twenty signals is regarded as a practical maximum.

In order to obtain a system that will require fewer memory elements and that will remember signals over a greater number of samples, another method of integration in which quantized signals are added has come into use. While it is possible to quantize the signal amplitude into many discrete levels, the particular type of quantized signal integration to be discussed in this report is one in which the signals are quantized into two amplitude levels and, of course, are quantized in time between fixed time markers. In the process of quantizing, if the complex signal and noise waveform between given time markers exceeds a predetermined amplitude, a standard pulse is generated at the end of the interval; if the threshold is not exceeded, no pulse is generated. The probability of obtaining a standard pulse can then be determined from the probability distribution function for the complex waveform in question.

For example, the signal plus noise prior to quantizing might have the probability density shown in Fig. 1. If the threshold is set as indicated and the cumulative probability calculated, the new distribution function that characterizes the quantized video consists of two lines of magnitude P at 1 (pulse present) and magnitude $1 - P$ at 0 (pulse absent).

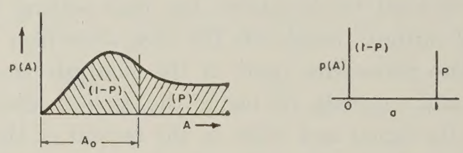

Fig. 1—Signal probability densities before and after quantization.

The method of integration then becomes a process of adding or counting a set of standard pulses with certain occurrence probabilities, instead of a process of superimposing and adding signal waveforms in successive repetition intervals. Some other obvious advantages result from the employment of quantized video, as in computer use, switching and memory circuits composed solely of bistable elements can be used to sort and count these pulses. Somewhat greater reliability is potentially attainable over the analog integration techniques, and, with the required circuitry reduced to an array of bistable elements, the use of transistors becomes quite attractive.

One of the first questions asked is that of the relative efficiency of binary integration as compared to analog integration. It is the purpose of this report to analyze the performance of the binary integration scheme, to indicate the threshold settings for best signal detection, and to compare the process with analog integration, indicating the relative merits and limitations of each method.

Basis of Analysis

A simplified block diagram of a general binary integration system is shown in Fig. 2 (next page). Signals coming from the second detector of the radar receiver must pass first through a threshold detector which quantizes the signal in amplitude and then through a quantizer which performs the same function in time. The quantized video

* The research in this document was supported jointly by the Army, Navy, and Air Force under contract with the Massachusetts Institute of Technology, Cambridge, Mass.

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[1] J. V. Harrington and T. F. Rogers, "Signal-to-noise improvement through integration in a storage tube," Proc. I.R.E., vol. 38, pp. 1197–1203; October, 1950. (See also references contained therein.)

is then gated in range so that pulses occurring in a given range interval are switched into a binary counter where they are counted over the number of repetition intervals in which the signal is expected to repeat. At the end of this period, the binary counter is sampled, and, if the number in the counter is some specified fraction of the total number of samples taken, a signal is judged to be present. Immediately, two questions concerning the operation of this integrator arise: first, how far down into the noise level should the input threshold be set in order to give good detection probabilities on weak signals without flooding the integrator with signals due to noise; second, what should be the minimum count in the counter over a given number of samples in order that the probability of detecting a signal, and not a noise pulse, shall be reasonably high.

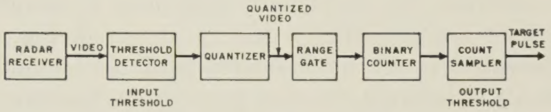

Fig. 2—Block diagram of binary integrator.

These two questions cannot be answered independently, and it is desired to determine the best setting for both input and output thresholds for best detection of small signals. The procedure used in the analysis is straightforward, and consists of using the known distribution functions for signal and noise in the output of the second detector in order to determine the occurrence probability for the quantized video pulse. From these, the discrete density functions for the sum of a given number of samples can be determined; and, finally, from these the detection probabilities following binary integration can be determined for noise alone and for signal plus noise. This procedure is followed for given input signal-to-noise ratios, for given settings of the input and output thresholds, and for integration of a given number of samples.

In order to keep the analysis simple and yet to obtain some useful results, it will be assumed that the pulses being counted will have uniform probabilities of occurrence. This immediately allows the use of the well-known binomial distribution and produces, in a straightforward fashion, a fairly useful result. It is implied in this assumption that all the samples being added have the same signal-to-noise ratio out of the second detector; this, in turn, implies that, in the radar case, the antenna is stationary. While this assumption of uniform occurrence probability is certainly not justified for a scanning radar, it does give a first approximation to the actual envelope of the signals, and the results can be subsequently modified to indicate the effect of a scanning antenna.

At this point, in order to get some physical feeling for the problem being analyzed, it will probably be helpful to refer to the set of quantized radar signals shown in Fig. 3. The picture shows successive repetition intervals of quantized radar video in the first 32 miles of range; here, the dots represent detected signals in a given half-mile range interval. The total spread of the picture is of the order of 3 degrees, or several beamwidths of the radar used, and a sequence of dots at constant range represents the occurrence of the target. The densely occupied region

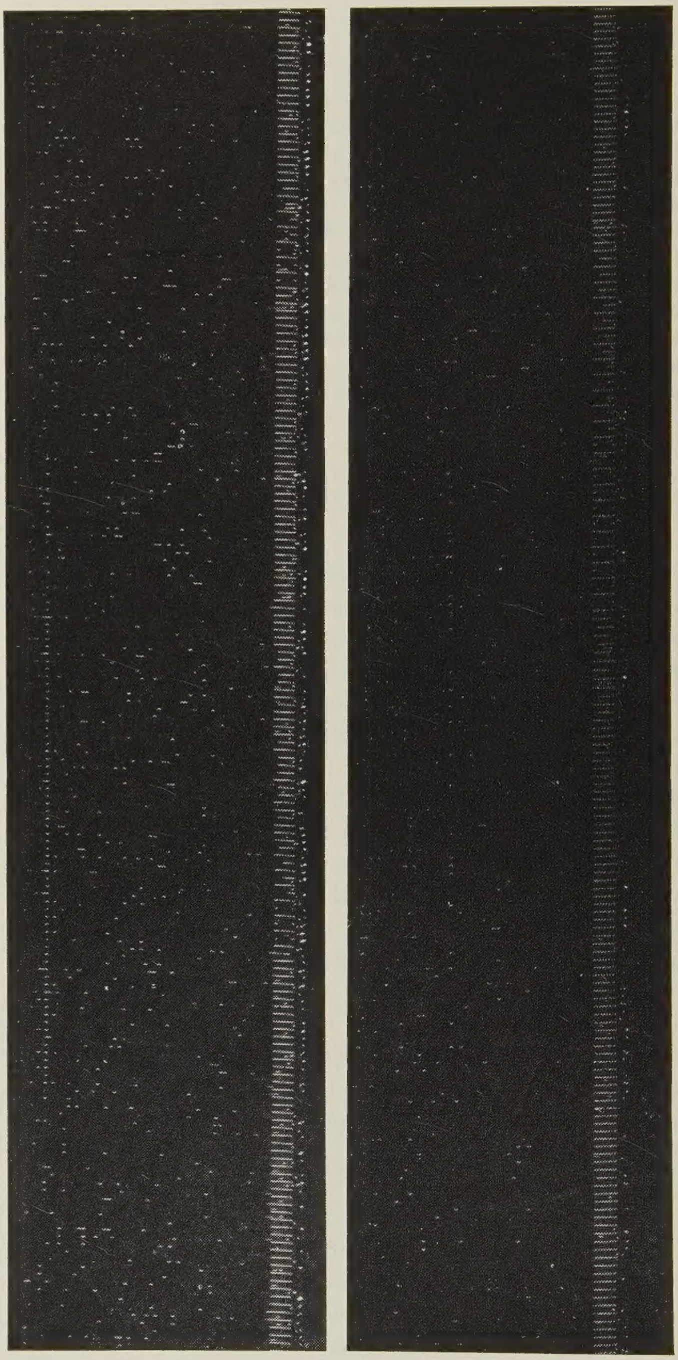

Fig. 3—Quantized video pattern taken over 32 miles in range and 4 degrees in azimuth.

at short ranges is characteristic of ground clutter, and the random dots throughout the field of the picture are due to receiver noise. What we are attempting to do is to determine, from the number of pulses in any one range interval and in any one beamwidth, whether a signal is present, or if noise alone is present.

Analysis

Fundamental Relationships

The probability density for the envelope R of a sine wave plus noise ($P \sin\omega t + I_n$), where the noise is confined to a relatively narrow band centered on the sine-wave frequency, is given by Rice as:[2]

$$p(R) = \frac{R\,dR}{\sqrt{\psi_0}} \exp\left\{-\frac{R^2 + P^2}{2\psi_0}\right\} I_0\left(\frac{RP}{\psi_0}\right), \qquad (1)$$

where ψ_0 is the noise power and I_0 is a modified Bessel function of the first kind and zero order.

For a linear second detector, the density functions for the peak amplitudes of the detected pulsed signals in noise are also given by (1). For the more general case of half-wave detection, R can be replaced by the appropriate $y = f(R)$—for example, $y = R^2$ for a square-law detector—to obtain the desired density functions.[3]

When the quantizing in time corresponds roughly to a pulse width, the probability of obtaining a quantized video pulse, given a pulse of amplitude P in the IF amplifier, is

$$P_s = P(a, v) = \int_V^\infty v\,dv \exp\left\{\frac{-v^2 + a^2}{2}\right\} I_0(av), \qquad (2)$$

where the variables have been normalized with respect to the noise power, i.e., $v = R/\sqrt{\psi_0}$, the predetection signal-to-noise ratio $a = P/\sqrt{\psi_0}$, and V is the normalized amplitude threshold in the quantizer.

In the special case where a signal is not present ($a = 0$), the probability of obtaining a quantized video pulse due to noise alone is

$$P_N = P(0, v) = \int_V^\infty v\,dv \exp\left\{\frac{-v^2}{2}\right\} = \exp\left\{\frac{-v^2}{2}\right\}. \qquad (3)$$

Having obtained the occurrence probabilities for quantized video pulses, the next step is to obtain the probability density for the sum of a given number of such pulses, that is, we ask the question, "What is the probability of obtaining exactly k successes (quantized pulses) in a set of m trials (number of repetition intervals over which we are integrating) when the probability of success in any one trial is p (P_s or P_n in our case)?" This is essentially the definition of the well-known binomial distribution[4] which is given by

$$b(m, k, p) = \frac{m!}{k!(m-k)!} p^k (1-p)^{m-k}. \qquad (4)$$

This then gives the probability of a counter reading of k out of a maximum counter reading of m. The probability of obtaining a counter reading of k or more is simply

$$Q(k, m, p) = \sum_k^m b(k, m, p). \qquad (5)$$

It can be demonstrated that the summation (5) is expressable in terms of the normalized, incomplete beta function which is tabulated.[5] Pearson's definition of this function is

$$I_x(p, q) = \frac{\int_0^x x^{p-1}(1-x)^{q-1}\,dx}{\int_0^1 x^{p-1}(1-x)^{q-1}\,dx}.$$

In our case,[6]

$$Q(k, m, p) = I_{1-p}(m - k + 1, k).$$

These discrete distributions, while describing rigorously the statistics of binary integrations, unfortunately do not lend themselves too well to calculation. It is more convenient to use the Edgeworth series approximation to the binomial distribution and then to obtain Q by integrating the approximate continuous distribution. Thus, the series that closely approximates $b(m, k, p)$ is[7]

$$P_m(y) = \left[\phi(y) - \frac{\alpha_3}{3!}\phi^{(3)}(y) + \frac{\alpha_4 - 3}{4!}\phi^{(4)}(y) + \frac{10}{\sigma!}\alpha_3^2\phi^{(6)}(y) + \cdots\right], \qquad (6)$$

where

$$y = \frac{k - \bar{m}}{\sigma}$$

$$\left.\begin{aligned} \bar{m} &= \text{the mean of the distribution} = mp \\ \sigma &= \text{the standard deviation} = \sqrt{mp(1-p)} \\ \alpha_3 &= \text{coefficient of skewness} = \frac{1-2p}{\sqrt{mp(1-p)}} \end{aligned}\right\} \qquad (6a)$$

$$(\alpha_4 - 3) = \text{coeficient of peakedness} = \frac{6p^2 - 6p + 1}{mp(1-p)}$$

and

$$\phi(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}, \qquad \phi^{(n)}(y) = \frac{d^n}{dy^n}\phi(y).$$

The desired cumulative probabilities may then be obtained by a termwise integration of (6), or

$$Q(k, m, p) = \int_Y^\infty P_m(y)\,dy, \qquad (7)$$

where $Y = k - \bar{m} - \frac{1}{2}/\sigma$.

[2] S. O. Rice, "Mathematical analysis of random noise," *Bell Sys. Tech. Jour.*, vol. 23, pp. 282–332; July, 1944; vol. 24, pp. 46–156; January, 1945.

[3] D. Middleton, "Some general results on the theory of noise through nonlinear devices," *Quart. Appl. Math.*, vol. 5, p. 445; 1948.

[4] W. Feller, "Probability Theory and its Applications," John Wiley and Sons, Inc., New York, N. Y., vol. I; 1950.

[5] K. Pearson, "Tables of the Incomplete Beta Function," The University Press, Cambridge, England; 1934.

[6] R. M. Fano, "The Transmission of Information—II," Tech. Rep. No. 149, Res. Lab. Elec., M. I. T., p. 21; February, 1950.

[7] T. C. Fry, "Probability and Its Engineering Uses," D. Van Nostrand Co., New York, N. Y., p. 256; 1928.

$$= \left[1 - \phi^{-1}(Y) - \frac{\alpha_3}{3!}\phi^{(2)}(Y) + \frac{\alpha_{4-3}}{4!}\phi^{(3)}(Y) + \frac{10}{6!}\alpha_3^2\phi^{(5)}(Y) \cdots \right],$$

where

$$\phi^{(-1)}(Y) = \int_{-\infty}^{Y} \phi(y)\, dy,$$

and is a tabulated function along with $\phi(y)$ and its first few derivatives.[8] The factor $\frac{1}{2}$ in the lower limit arises by virtue of its allowing a better agreement between the cumulative probabilities determined from the discrete and from the continuous density functions.[4]

A comparison of the results obtained from (5) and (7) is shown in Fig. 4 where the curves were obtained from tables[5] of the incomplete beta function and the plotted points were obtained by using the indicated number of terms of (7). For large m, the density function (6) implies that the distribution tends toward a normal distribution; it is evident from the curves, especially for small p and k, that the higher-order terms in (7) must be considered. For large p fit with normal distribution is quite good.

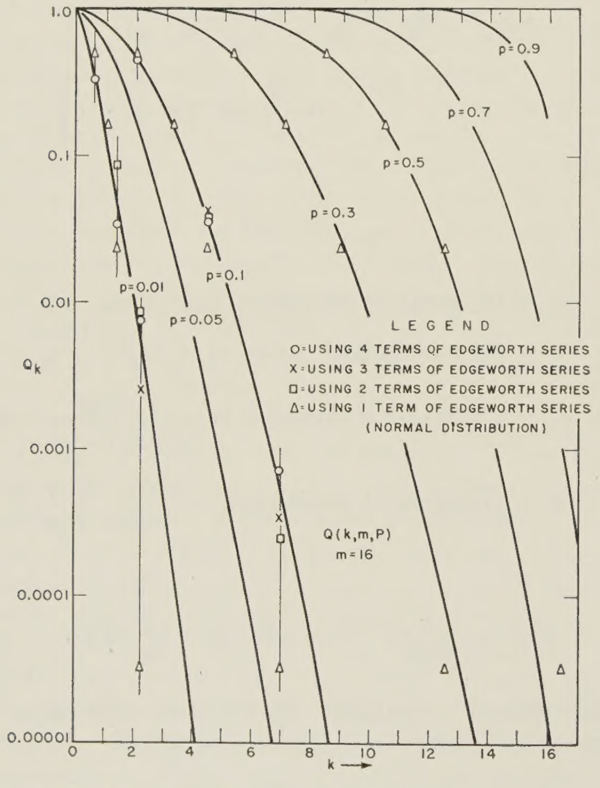

Fig. 4—Plot of Q_k vs k for various numbers of terms and values of p.

We have, then, all the mathematical apparatus on hand for calculating the detection probabilities, following integration, for signal plus noise, and for noise alone, as functions of the quantizer threshold V and the counter threshold K. To study the effects of these thresholds on the detection efficiency, it is convenient to define a signal-to-noise ratio for the integrated signals and to maximize this with respect to V.

The signal-to-noise ratio for signals and noise that have different probability distributions is, in general, not too significant a quantity from which detection probabilities can be deduced. This is so because a simple ratio cannot, by itself, specify the two distributions, and hence cannot specify the cumulative probabilities derived from them. This is unlike the situation in, say, an IF amplifier where signals and noise are linearly superimposed so that the distribution of noise on the signal is the same for any signal amplitude including 0. In this case, a signal-to-noise ratio does completely specify the two distributions for signals plus noise and noise alone. This predetector signal-to-noise ratio a is, consequently, a very significant quantity which is referred to throughout the analysis.

Nevertheless, for purposes of expediency in analysis, it is desirable to define a signal-to-noise ratio for detected video. Where a reasonable amount of integration is used, such that the distributions approach the normal distribution, the signal-to-noise ratio can have appreciable utility. It cannot completely specify the desired detection probabilities, but from it a good approximation to these quantities can be obtained.

While there are several ways in which one could define a signal-to-noise ratio for detected signals in noise, the particular definition that is most significant in our case is the following. The signal-to-noise ratio of the integrated video will be defined as the difference between the means of the distribution for signal plus noise and noise alone divided by the standard deviation of the integrated noise alone. Thus, from (6a)

$$(S/N)_{\text{out}} = \frac{mP_s - mP_N}{\sqrt{mP_N(1 - P_N)}} = \sqrt{m}\,\frac{P_s - P_N}{\sqrt{P_N(1 - P_N)}} = \sqrt{m}\,\rho, \qquad (8)$$

where

$$\rho = \frac{P_s - P_N}{\sqrt{P_N(1 - P_N)}}.$$

From this it is evident that

$$\frac{P_s - P_N}{\sqrt{P_N(1 - P_N)}}$$

is the equivalent signal-to-noise ratio ρ of the quantized video and that this is improved by the well-known square root of the number of samples taken. It is interesting to note that ρ can be determined directly from the discrete distribution of Fig. 1, that is, the corresponding mean for signal present is P_s and that for noise P_N, while the standard deviation for the noise is $\sqrt{P_N(1 - P_N)}$.

The Effects of Quantization on Signal-to-Noise Ratio

To determine the variation of ρ with P_N, it is first necessary to obtain an expression for P_s as a function of

[8] "Tables of the Error Function and Its First Twenty Derivatives," *Annals Harvard Computation Lab.*; January, 1922.

P_N, that is, since P_s and P_N are both functions of V (the input threshold), it should be possible to eliminate V and obtain $P_s = f(a, P_N)$.

By making the substitution

$$p = e^{-v^2/2}$$

in (2), and remembering that

$$P_N = e^{-V^2/2},$$

we obtain

$$P_s = 1 - e^{-a^2/2} \int_{P_N}^{1} I_0(a\sqrt{-2 \log p})\, dp. \qquad (9)$$

Since (9) implies that the relationship between P_s and P_N is independent of V, it must also be independent of any function of V that might characterize a different detector. Thus, we reach the interesting and very useful conclusion that the P_s vs P_N *relationship is independent of the detector characteristic,* and is a consequence solely of the process of half-wave rectification and filtering without noticeable video narrowing. Our results will then apply for any half-wave detector of the form $Z = f(v)$ for $v > 0$.

Some useful series expansion of (9) may be obtained by substituting the series expansion for I_0 and integrating termwise; thus

$$I_0(x) = 1 + \sum_{k=1}^{\infty} \frac{x^{2k}}{2^2 \cdot 4^2 \cdots (2k)^2}, \qquad (x < \infty),$$

and

$$I_0(a\sqrt{-2 \log np}) = 1 + \sum_{k=1}^{\infty} \frac{a^{2k}(-\log np)^k}{2^k(k!)^2}, \qquad (p > 0),$$

using the relationship

$$\int \log^k p\, dp = \sum_{n=0}^{k} (-1)^{k-n} \frac{k!}{n!} p \log^n p.$$

After integrating and collecting terms, we find

$$\begin{aligned} P_s = 1 - e^{-a^2/2}(1 - P_N) & \\ - \frac{a^2}{2} e^{-a^2/2}(1 - P_N - P_N(-\log P_N)) - \frac{1}{2!} & \\ \cdot\left(\frac{a^2}{2}\right) e^{-a^2/2}\Big(1 - P_N - P_N(-\log P_N) & \\ - \frac{1}{2!} P_N(-\log P_N)^2\Big) \cdots, & \qquad (10) \end{aligned}$$

or

$$P_s = 1 - e^{-a^2/2} \sum_{n=0}^{\infty} \frac{1}{n!} \lambda_n \left(\frac{a^2}{2}\right)^n,$$

where

$$\lambda_n = \left[1 - \sum_{k=0}^{n} \frac{1}{k!} P_N(-\log P_N)^k\right].$$

The coefficients λ_n are functions only of P_N. It may be shown that λ_n must be less than unity, i.e., $0 \leq \lambda_n \leq 1$. For large n, $\lambda_n \rightarrow 0$, for all P_N, so that the series converges fairly rapidly.

An alternative expression for P_s is obtainable by regrouping the terms in powers of $(-\log P_N)$; thus

$$P_s = P_N + (1 - e^{-a^2/2})P_N(-\log P_N) + \frac{1}{2!}\left(1 - e^{-a^2/2} - \frac{a^2}{2} e^{-a^2/2}\right) P_N(-\log P_N)^2 + \cdots, \qquad (11)$$

or

$$P_s = P_N + \sum_{n=1}^{\infty} \frac{\mu_n}{n!} P_N(-\log P_N)^n,$$

where

$$\mu_n = 1 - e^{-a^2/2} \sum_{k=0}^{n-1} \frac{1}{k!}\left(\frac{a^2}{2}\right)^k;$$

the coefficients μ_n are functions only of a. Also $0 \leq \mu_n \leq 1$ and, for large n, $\mu_n \rightarrow 0$ for all $a < \infty$, which again indicates fairly rapid convergence.

Still another form of (10), particularly useful for the small-signal case, is

$$P_s = P_N + \frac{a^2}{2} e^{-a^2/2} P_N(-\log P_N) + \frac{1}{2!}\left(\frac{a^2}{2}\right)^2 e^{-a^2/2} \cdot\left(P_N(-\log P_N) + \frac{1}{2!} P_N(-\log P_N)^2\right) + \cdots. \qquad (12)$$

An expression for ρ can then be obtained by using any of these and (8). Thus

$$\rho = \sqrt{\frac{1 - P_N}{P_N}} - \frac{e^{-a^2/2}}{\sqrt{P_N(1 - P_N)}} \sum_{n=0}^{\infty} \frac{\lambda_n}{n!}\left(\frac{a^2}{2}\right)^n \qquad (13)$$

or

$$\rho = \frac{1}{\sqrt{P_N(1 - P_N)}} \sum_{n=1}^{\infty} \frac{\mu_n}{n!} P_N(-\log P_N)^n. \qquad (14)$$

These are very useful and rigorous series for the effective signal-to-noise ratio of binary quantized signals. The expansions are made in terms of the predetector signal-to-noise ratio and the quantizer threshold as specified by the probability of quantizing noise. Furthermore, they hold for any half-wave envelope detector. The variation of ρ

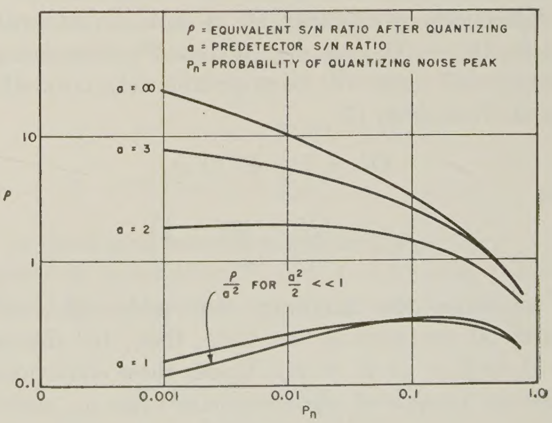

Fig. 5—ρ vs P_n for constant α.

with P_N for various a is shown in Fig. 5. It is evident from these curves that there is an optimum setting of the input threshold (i.e., optimum P_N) for any given a to give a maximum ρ. For $a = 1$, which ordinarily corresponds to

the minimum detectable signal in any practical post-detector integration system, the optimum $P_N = 0.1$. For larger a, the optimum P_N decreases, but P_N should be set for best results on the weakest signal of interest. It is further evident from the curves that, for large a, ρ does not increase indefinitely, but reaches a limiting value which is a function of P_N, and is given by the first term in (13). It is also obtainable by realizing that, for large a, $P_s \rightarrow 1$, in which case (8) becomes

$$\rho = \sqrt{\frac{1 - P_N}{P_N}} \qquad \text{for} \qquad a \gg 1. \tag{15}$$

This saturation effect on ρ is not a serious limitation in our case since, for $P_N = 0.1$, $\rho_{\max} = 3$, which is sufficiently high to insure good detection even without integration.

For the small-signal case, i.e., $a^2/2 \ll 1$, the first few terms of (12) give a good approximation to P_s. Thus

$$P_s \simeq P_N + \left(\frac{a^2}{2}\right) P_N(-\log P_N) \qquad \text{for} \qquad \frac{a^2}{2} \ll 1. \tag{16}$$

Hence

$$\rho \simeq \frac{a^2}{2}\sqrt{\frac{P_N}{1 - P_N}}\,(-\log P_N) \qquad \text{for} \qquad \frac{a^2}{2} \ll 1. \tag{17}$$

Differentiation of this, with respect to P_N, indicates a maximum at $P_N = 0.203$. Hence, for very small signals,

$$\rho_{\max} = \frac{a^2}{2}(0.804) \qquad \text{at} \qquad P_N = 0.203. \tag{18}$$

We conclude then that the optimum setting for P_N approaches 0.203 as a diminishes. A plot of (17) is shown in Fig. 5 and the agreement between it and the accurate contour for $a = 1$ is quite good.

Determination of Best Counter Threshold K

Having shown that, for best weak-signal detection, P_N should be set for 0.1—i.e., $a = 1$ is ordinarily the weakest signal detectable with a reasonable number of hits per beamwidth—the problem is to select K so that the allowable false-alarm probability Q_N is met. An approximate figure for $Q_N = (Q_N^1)$ can be calculated by assuming that the integrated noise will be approximately normally distributed. Thus from (7)

$$Q_N^1 = 1 - \phi^{-1}(Y_N), \tag{19}$$

where

$$Y_N = \left(\frac{K - \bar{m}_N - \frac{1}{2}}{\sigma_N}\right).$$

If we define the minimum detectable signal as one detected 50 per cent of the time, then, for the signal, $Y_s = 0$, or $K - (1/2) = \bar{m}_s$. Under these conditions, Y_N signifies an integrated signal-to-noise ratio ρ_{out} since

$$Y_N = \frac{\bar{m}_s - \bar{m}_N}{\sigma_N} = \sqrt{m}\cdot\rho = \rho_{\text{out}}. \tag{20}$$

For a desired false-alarm probability Q_N, the tables yield a corresponding value for Y_N, i.e., if $Y_N = 4$, $Q_N^1 = 3 \times 10^{-5}$. Then

$$(K - \tfrac{1}{2}) = (\rho_{\text{out}})\sqrt{mP_N(1 - P_N)} + mP_N. \tag{21}$$

For $P_N = 0.1$, this reduces to

$$K = (\rho_{\text{out}})0.3\sqrt{m} + 0.1m + \tfrac{1}{2}, \tag{21}$$

from which Table I, for best counter threshold settings, is obtained.

TABLE I

m	$\rho_{\text{out}} = 4$ $Q_N = 3 \times 10^{-5}$	$\rho_{\text{out}} = 5$ $Q_N < 10^{-6}$
4	3.3	3.9
8	4.7	5.6
16	6.9	8.1
32	10.5	12.2
64	16.5	18.9

For convenience when using a binary counter, these would ordinarily be rounded off to the nearest power of 2, incurring a very slight loss in sensitivity.

The precise false-alarm probability Q_N can be determined from (7), which for $P_N = 0.1$ becomes

$$Q_N = 1 - \phi^{(-1)}(\rho_{\text{out}}) - \frac{1}{3!}\left(\frac{8}{3\sqrt{m}}\right)\phi^{(2)}(\rho_{\text{out}}) + \frac{1}{4!}\left(\frac{46}{9}\right)$$
$$\cdot\frac{1}{m}\phi^{(3)}(\rho_{\text{out}}) + \frac{10}{6!}\left(\frac{8}{3}\right)^2\frac{1}{m}\phi^{(5)}(\rho_{\text{out}}) + \cdots. \tag{22}$$

With ρ_{out} specified, then for a given number of integrations m the value of ρ is also specified, which by the relationships given previously yields a value for $a_{\min}$, the minimum detectable signal. The curves of Fig. 6 indicate the variation of $a_{\min}$ with m (the number of samples integrated). It will be noted that the minimum detectable signal decreases very slowly with an increase in the number of integrations since $a_{\min} \propto (1/4\sqrt{m})$. This fourth-root relationship for small signals follows from (17) and (8) since $\rho_{\text{out}} = \sqrt{m}\rho$, and $\rho \propto a^2$ then for constant ρ, $a \propto (1/4\sqrt{m})$.

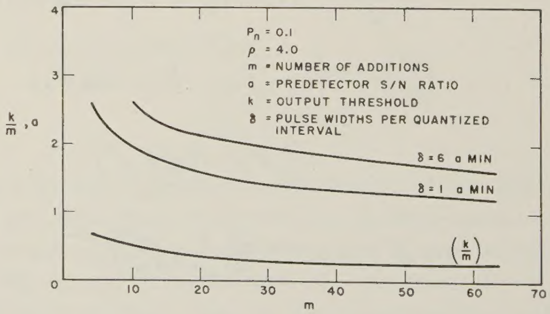

Fig. 6—k/m and minimum α, vs m.

Comparison with Analog Integration

A comparison of binary integration with analog integration can be reduced to a comparison of the effective signal-to-noise ratios prior to integration in the two cases,

since for both cases the effect of integration is to multiply this input signal-to-noise ratio by $\sqrt{m}$. The two must be defined, however, in the same manner, and we shall define the signal-to-noise ratio for the continuous distributions dealt with in the analog case in the same manner as we did for the quantized case—namely, as the difference between the means of the signal and noise distributions divided by the standard deviation of the noise alone. It may be shown that, for the continuous distributions, the postdetector signal-to-noise ratio $\bar{m}/\sigma(0)$ is given[1,3] by

$$\frac{\bar{m}}{\sigma_{(0)}} = \frac{1}{\sqrt{\dfrac{\Gamma(1+k)}{\Gamma^2\left(1+\dfrac{k}{2}\right)} - 1}} \left[{}_1F_1\left(-\frac{k}{2}, 1, -\frac{a}{2}\right)^2 - 1 \right], \qquad (23)$$

where k is the power of the detector law and ${}_1F_1$ is the confluent hyper-geometric function.[9]

For the small-signal case, $a^2/2 \ll 1$, this becomes

$$\frac{\bar{m}}{\sigma_{(0)}} \simeq \frac{\dfrac{k}{2}}{\sqrt{\dfrac{\Gamma(1+k)}{\Gamma^2\left(1+\dfrac{k}{2}\right)} - 1}} \cdot \frac{a^2}{2}, \qquad (24)$$

which, for the linear detector ($k = 1$), reduces to

$$\frac{\bar{m}}{\sigma_{(0)}} \simeq 0.960\left(\frac{a^2}{2}\right). \qquad (25)$$

For the square-law detector ($k = 2$), (23) reduces without approximation to

$$\frac{\bar{m}}{\sigma_{(0)}} = \frac{a^2}{2}. \qquad (26)$$

In general, the coefficient of $(a^2/2)$ in (24) has a maximum value of unity for $k = 2$ and falls off slowly for all other values of k.

In the small-signal case for quantized signals, we recall from (18) that

$$\rho = 0.804\left(\frac{a^2}{2}\right). \qquad (18)$$

A comparison of (18) with (25) and (26) leads to the conclusion that the effect of quantization on the signal-to-noise ratio is to reduce it by a factor 0.838 or 1.54 db in comparison with the signal-to-noise ratio from a linear detector, and 0.804 or 1.92 db in comparison with that from a square-law detector. Since the square-law detector yields the highest signal-to-noise ratio for small signals, the maximum loss sustained is of the order of 1.9 db. It should be remembered, however, that most analog integrators are nonideal, i.e., the signals decay in the memory, and losses of 1 to 2 db in output signal-to-noise ratio from this cause are not at all uncommon in practical devices of this type. It may be stated, therefore, that, while analog integration starts off with an advantage of 1.5 to 2 db in input signal-to-noise ratio, the perfect memory of the binary integrator makes up for this loss and the two systems should be about equivalent in final detection efficiency.

[9] D. Middleton and V. Johnson, "A Tabulation of Selected Confluent Hypergeometric Functions," Tech. Rep. No. 140, Cruft Labs., Harvard University; January, 1952.

A comparison of ρ and $(\bar{m}/^{\sigma}(0))$ for the linear detector is shown in Fig. 7. In the weak-signal region, $a < 2$, the difference is small. Both, of course, suffer markedly from the small-signal suppression effect—i.e., signal-to-noise αa^2 for small signals—as a comparison with the predetector signal-to-noise ratio a will demonstrate.

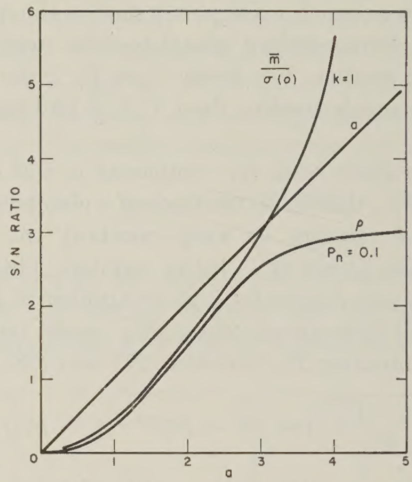

Fig. 7—Comparison of predetector, postdetector and postquantizer signal-to-noise ratios.

The Effect of the Width of Quantizing Interval

The previous analysis was based on having either a signal pulse or a noise pulse in the quantizing interval. Now we turn to the more usual case wherein the quantizing interval is several, say, δ pulse widths in duration. If independence is assumed between the signal levels one pulsewidth apart (an assumption that the usual match between circuit bandwidth and pulsewidth makes reasonable), then the occurrence probabilities for the quantized signals and noise P_s' and P_N' are related to their former values by

$$(1 - P_s') = (1 - P_s)(1 - P_N)^{\delta - 1}, \qquad (27)$$
$$(1 - P_N') = (1 - P_N)^{\delta},$$

where, in the signal case, it is assumed that there is one signal pulse of probability P_s and $(\delta - 1)$ noise pulses of probability P_N within the quantizing interval. As in the earlier case,

$$\rho_\delta = \frac{P_s' - P_N'}{\sqrt{P_N'(1 - P_N')}}. \qquad (28)$$

A comparison of ρ_δ with ρ_1 can be made on many bases; and the easiest, although not the most significant, comparison to make is on the basis of constant P_N, i.e., same input threshold setting, independent of δ. From (27) and (28)

$$\rho_\delta = \frac{P_s - P_N}{\sqrt{P_N(1-P_N)}} \cdot \sqrt{\frac{P_N(1-P_N)^{\delta-1}}{1-(1-P_N)^\delta}}$$

$$= \rho_1 \cdot \sqrt{\frac{P_N(1-P_N)^{\delta-1}}{1-(1-P_N)^\delta}}. \tag{29}$$

For small P_N such that $\delta P_N \ll 1$, this becomes

$$\rho_\delta \sim \rho_1 \frac{1}{\sqrt{\delta}}. \tag{30}$$

This result agrees with the intuitive feeling that, since there is δ times as much noise power now in the quantized interval, the corresponding signal-to-noise ratio should be $\sqrt{\delta}$ times smaller. For larger δ or P_N, the ratio of ρ_δ/ρ_1 becomes much smaller than $1/\sqrt{\delta}$ and approaches 0 as $P_N \to 1$.

The more realistic basis for comparing ρ_δ and ρ_1 is that of a constant P_N', that is, on the basis of a changing threshold (P_N) as δ changes to keep constant the average number of noise pulses (P_N') being counted. This is ordinarily the way any system for a given number of additions and a specified false-alarm probability would have to be adjusted. Eliminating P_N between (27) and (28), we find

$$\rho_\delta = \sqrt{\frac{1-P_N'}{P_N'}}\,[1-(1-P_N')^{-1/\delta}(1-P_s)], \tag{31}$$

where P_s is given by (10), (11), or (12) with

$$P_N = (1-(1-P_N')^{1/\delta}).$$

For large signals (large a), $P_s \to 1$, and

$$\rho_\delta \simeq \sqrt{\frac{1-P_N'}{P_N'}}, \tag{32}$$

which is precisely the same result we obtained for ρ_1.

In the small-signal case, (31) reduces to

$$\rho_\delta = \frac{a^2}{2}\sqrt{\frac{(1-P_N')^{\delta-2/\delta}}{P_N'}}$$

$$\cdot\{[1-(1-P_N')^{1/\delta}][-\log(1-(1-P_N'))]^{1/\delta}\};$$

$$\frac{a^2}{2} \ll 1. \tag{33}$$

A further simplification of (33) can be made for small P_N'. Thus, if $P_N'/\delta \ll 1$, then

$$\rho_\delta \simeq \frac{a^2}{2}\sqrt{\frac{1-P_N'}{P_N'}}\left[\frac{\frac{P_N'}{\sigma}\;\frac{-\log P_N'}{\sigma}}{1-\frac{P_N'}{\delta}}\right], \qquad \frac{a^2}{2} \ll 1. \tag{34}$$

The variation of ρ_δ with P_N' as calculated from (33) for various values of δ is plotted in Fig. 8. The general nature of all these curves is much the same; the magnitude of ρ_δ, of course, reduces for increasing δ, and the maxima shift toward higher P_N. For $\delta = 6$ and $P_N' = 0.1$, the ratio of (ρ_1/ρ_δ) is 3.44. However this does not mean that the minimum detectable a is 3.44 times smaller when $\delta = 6$; it means that, for the same predetector signal-to-noise ratio a, the quantized signal-to-noise ratio is that much poorer. Because of the detector loss, however, wherein $\rho \propto a^2$, the ratio of the minimum detectable signals for $\delta = 1$, and for $\delta = 6$, all other things being equal, is $\sqrt{3.44}$ or 1.85.

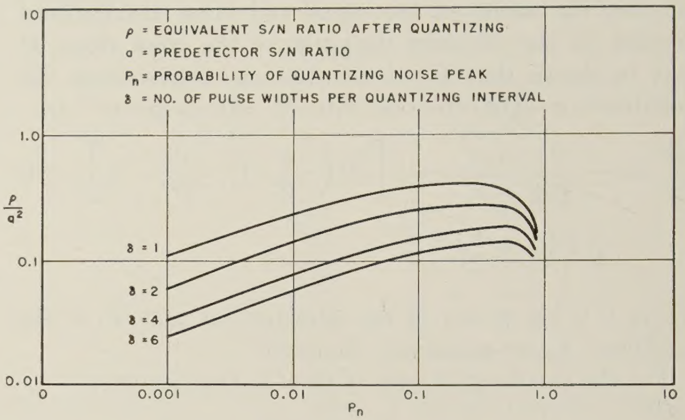

Fig. 8—ρ vs P_n for various α, $\alpha^2/2 \ll 1$.

Effective Signal-to-Noise Ratio for Signals of Varying Amplitudes

When the signals to be integrated are derived from a radar with a scanning antenna, their amplitude is not constant as assumed in the previous analysis, but varies with time [$a = a(t)$]. On the average, the envelope of the set of signals from a given target will resemble the beam pattern of the antenna, and the occurrence probability for the corresponding quantized video pulses will vary accordingly. The mean of the integrated signal distribution, instead of being mPs as before, will now be given by

$$\overline{m}_s = \sum_{j=1}^{m} P_{s_j} = m\overline{P}_s$$

where

$$\overline{P}_s = \frac{1}{m}\sum_{j=1}^{m} P_{s_j}. \tag{35}$$

From (10) and (13), therefore

$$\rho = \sqrt{\frac{1-P_N}{P_N}} - \frac{1}{\sqrt{P_N(1-P_N)}}\sum_{n=0}^{\infty}\frac{\lambda_n}{n!}\cdot\frac{1}{m}\sum_{j=1}^{m}\left(\frac{a_j^2}{2}\right)^n e^{-a^2/2}, \tag{36}$$

where $a_j = a(t_j)$ is the predetector signal-to-noise ratio of the jth sample in the set of m being counted. Eq. (36), then, is the desired expression for the quantized signal-to-noise ratio in the case of nonuniform signal amplitudes. It is, however, a rather cumbersome equation to use, and fortunately in the small-signal case (36) simplifies to:

$$\rho \simeq \frac{\overline{a^2}}{2}\sqrt{\frac{P_N}{1-P_N}}(-\log P_N), \quad \text{for} \quad \frac{\overline{a^2}}{2} \ll 1, \tag{37}$$

where

$$\overline{a^2} = \frac{1}{m}\sum_{j=1}^{m} a_j^2,$$

and is the mean square signal-to-noise ratio of the set of

m samples. The results given previously on counter thresholds and optimum P_N will still hold for small signals. To get the minimum detectable a, however, $\overline{a^2}$ instead of a^2 must be used in the varying-signal case.

Relationships to More-General Detection Theory

It should be mentioned here that this report describes an analysis of a particular method of signal detection where, for the sake of circuit economy, particular criteria are used to determine the presence or absence of a signal. The statistical criterion used is essentially the Neyman-Pearson one, and the question of whether or not this is the best possible criterion has not been considered.

The general theory of testing samples to determine whether they fall under hypothesis H_1 (in our case, noise) or under hypotheses H_2 (in our case, signal) is one that has received a great deal of attention in the field of quality control and lately in the field of communications. In testing for hypothesis H_2, two types of errors are considered. These are: a Type 1 error, having probability α, in which H_2 is judged to be true when it is not (H_1 is true); and a Type 2 error, having probability B, in which H_2 is judged to be false when it is not (H_1 is false).

It may be rather loosely stated that the general object of the testing procedure is to minimize α and B—that is, to maximize the probability of being right. Various methods for examining an ensemble of samples to judge their nature have been proposed, and recently an extensive analysis has been made by Middleton of three such methods which he applied to the detection of signals in noise.[10] The criteria differ primarily in the manner in which α, B, and n (the number of samples) are varied or held constant for a given predetector signal-to-noise ratio a.

For instance, the Neyman-Pearson observer holds α and n constant and produces a variable B with varying a, while the sequential observer fixes α and B, and uses a varying number of samples n for varying a. In the case analyzed in this report, we considered integration of a fixed number of samples and adjusted the thresholds to obtain an acceptable false-alarm probability Q_N. The detection probability Q_s then varied with a and this, of course, is characteristic of the Neyman-Pearson observer. The relationships between α, B, and our Q_N, Q_s are simple, and are given by

$$\alpha = Q_N, \qquad B = 1 - Q_s. \tag{38}$$

In the discussion of these three observers, Middleton[10] employs the notion of a betting curve where the betting function is defined as

$$\omega(a^2) = p(1 - \alpha) + q(1 - B); \tag{39}$$

p and $q = (1 - p)$ are essentially arbitrary weighting factors which assess the importance that H_1 or H_2, respectively, are true. If, over a large number of tests, signals and noise are equally likely, and the determination of both is equally important, then $p = q = 1/2$. The betting function may be interpreted as the probability of making the correct decision no matter which of the two hypotheses actually obtains.

[10] D. Middleton, "Statistical Criteria for the Detection of Pulsed Carriers in Noise," AF Cambridge Res. Center Rep. No. E-5091; August, 1952.

In our analysis, we have derived expressions for Q_s and Q_N from which a curve of Q_s as a function of Q_N and a could be obtained. This would have the general character of that in Fig. 9. The entire curve has not been calculated, however, but we have confined our attention to two points on it. In a practical sense, since the curve is necessarily a monotonically increasing function, these two points pretty well specify the function. These points are Q_N at $a = 0$, and $Q_s = 0.5$ at $a = a_{\min}$ (III-A). The relationship between detection probability curve and betting curve is readily obtained from (38) and (39); thus

$$Q_s = \frac{\omega(a^2) - pQ_N}{q}. \tag{40}$$

Since p and Q_N are constants in our case, Q_s is linearly related to the betting function and the curve of Fig. 9 can be interpreted as a type of betting curve.

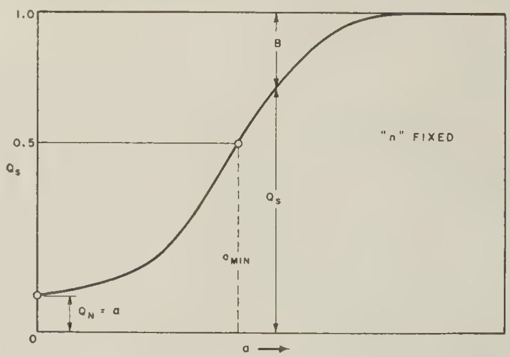

Fig. 9—Detection probability as a function of predetector signal-to-noise ratio.

While the question of which of the statistical criteria gives the "best" results has not been treated in this report, there is some evidence[10] to indicate that the Neyman-Pearson observer in the radar case performs about as well as and, in some cases, slightly better than the other observers. Hence, in addition to being the easiest criterion to implement, it is also likely to be one of the most sensitive. A possible reason for this is that the decision made is based on the total (integrated) received energy. In a physical sense, one intuitively feels that if the total signal energy is taken into account, the detection efficiency will not be significantly different in any of the statistical detection schemes.

Acknowledgment

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An Expansion for Some Second-Order Probability Distributions and its Application to Noise Problems*

J. F. BARRETT† AND D. G. LAMPARD‡

Summary—In this paper it is shown that, in general, second-order probability distributions may be expanded in a certain double series involving orthogonal polynomials associated with the corresponding first-order probability distributions. Attention is restricted to those second-order probability distributions which lead to a "diagonal" form for this expansion.

When such distributions are joint probability distributions for samples taken from a pair of time series, some interesting results can be demonstrated. For example, it is shown that if one of the time series undergoes an amplitude distortion in a time-varying "instantaneous" nonlinear device, the covariance function after distortion is simply proportional to that before distortion.

Some simple results concerning conditional expectations are given and an extension of a theorem, due to Doob, on stationary Markov processes is presented.

The relation between the "diagonal" expansion used in this paper and the Mercer expansion of the kernel of a certain linear homogeneous integral equation, is pointed out and in conclusion explicit expansions are given for three specific examples.

Introduction

The solution of most problems involving noise in nonlinear devices is difficult. However, a few years ago Bussgang[1] demonstrated an interesting and useful result. He showed that, if one of a pair of stationary time series with Gaussian probability distributions was amplitude distorted in a fixed "instantaneous" nonlinear device, then the cross-correlation function after the distortion is proportional to the cross-correlation function before the distortion.

An attempt to extend the scope of Bussgang's results to distributions other than Gaussian, led to a class of distributions which is discussed in this paper. Such distributions exhibit a number of interesting properties some of which are studied here. It is believed that many results will be useful in dealing with certain noise problems.

Analysis

We suppose that $p(x_1\,;\,x_2)$ is a second-order probability distribution. The corresponding first-order probability distributions are given by

$$\left.\begin{aligned} p_1(x_1) &= \int_{\text{w.r.}} p(x_1\,;\,x_2)\,dx_2 \\ p_2(x_2) &= \int_{\text{w.r.}} p(x_1\,;\,x_2)\,dx_1 \end{aligned}\right\}. \tag{1}$$

We construct[2] two sets of normalized orthogonal polynomials $\theta_n^{(1)}(x_1)$ and $\theta_n^{(2)}(x_2)$ such that

$$\left.\begin{aligned} \int_{\text{w.r.}} p_1(x_1)\,\theta_m^{(1)}(x_1)\,\theta_n^{(1)}(x_1)\,dx_1 &= \delta_{mn} \\ \int_{\text{w.r.}} p_2(x_2)\,\theta_m^{(2)}(x_2)\,\theta_n^{(2)}(x_2)\,dx_2 &= \delta_{mn} \end{aligned}\right\}. \tag{2}$$

We now expand the second-order probability distribution in a double "Fourier" series involving these polynomials. Thus

$$p(x_1\,;\,x_2) = p_1(x_1)p_2(x_2)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn}\,\theta_m^{(1)}(x_1)\,\theta_n^{(2)}(x_2). \tag{3}$$

The coefficients a_{mn} may be determined in the usual way by multiplying both sides by $\theta_k^{(1)}(x_1)\;\theta_l^{(2)}(x_2)$ and integrating using the orthogonal property (2). We find

$$a_{mn} = \iint_{\text{w.r.}} p(x_1\,;\,x_2)\,\theta_m^{(1)}(x_1)\,\theta_n^{(2)}(x_2)\,dx_1\,dx_2\,. \tag{4}$$

Thus we see from (3) that the second-order distribution is now determined completely by the two first-order distributions and the coefficient matrix $[a_{mn}]$.

In this paper we restrict our attention[3] to the class, Λ say, of distributions $p(x_1\,;\,x_2)$ which have the property that the matrix $[a_{mn}]$ is diagonal. As we shall see from examples, some important distributions that occur in physical problems are in this class.

Thus for all $p(x_1\,;\,x_2)$ in Λ, (3) may be written[4]

$$p(x_1\,;\,x_2) = p_1(x_1)p_2(x_2)\sum_{n=0}^{\infty} a_n\,\theta_n^{(1)}(x_1)\,\theta_n^{(2)}(x_2), \tag{5Λ}$$

where the coefficient a_n is now given by

$$a_n = \iint_{\text{w.r.}} p(x_1\,;\,x_2)\,\theta_n^{(1)}(x_1)\,\theta_n^{(2)}(x_2)\,dx_1\,dx_2\,. \tag{6Λ}$$

We now determine the form of the polynomials of degree 0 and 1. As $p_1(x_1)$ and $p_2(x_2)$ are probability distributions we must have

$$\int_{\text{w.r.}} p_1(x_1)\cdot 1\cdot 1\cdot dx_1 = 1 \tag{7}$$

* Some of the material of this paper is taken from a Dissertation submitted by D. G. Lampard for the Ph.D. degree at the University of Cambridge.

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[1] J. J. Bussgang, "Cross correlation Functions of Amplitude Distorted Gaussian Signals," Tech. Rep. No. 216, Res. Lab. Elec., MIT; March, 1952.

[2] By applying the Gram-Schmidt procedure to the sequence $1, x, x^2 \cdots$. For details see Courant and Hilbert, "Methods of Mathematical Physics," Interscience Publishers, Inc., New York, N. Y., vol. I, p. 50; 1953.

[3] The authors have not been able so far to find what general restrictions must be placed on $p(x_1\,;\,x_2)$ in order that it may belong to Λ.

[4] Throughout this paper, equations which are only true when $p(x_1\,;\,x_2)$ belongs to Λ, will be denoted by having Λ added after the equation number.

$$\int_{\text{w.r.}} p_1(x_1)\cdot(x_1 - \mu_1)\cdot 1\cdot dx_1 = 0 \qquad (8)$$

$$\int_{\text{w.r.}} p_1(x_1)(x_1 - \mu_1)(x_1 - \mu_1)\, dx_1 = \sigma_1^2\,, \qquad (9)$$

with corresponding results for $p_2(x_2)$. Here we have used the symbols μ, and σ_1 for the mean and standard deviation respectively. It follows that

$$\left.\begin{aligned} \theta_0^{(1)}(x_1) &= 1; & \theta_0^{(2)}(x_2) &= 1 \\ \theta_1^{(1)}(x_1) &= \frac{x_1 - \mu_1}{\sigma_1}\,; & \theta_1^{(2)}(x_2) &= \frac{x_2 - \mu_2}{\sigma_2} \end{aligned}\right\}. \qquad (10)$$

On using (10) and (6), it is easy to see that the coefficients a_0 and a_1 are just

$$a_0 = 1 \qquad (11)\Lambda$$

$$a_1 = \frac{\langle (x_1 - \mu_1)(x_2 - \mu_2)\rangle}{\sigma_1\sigma_2} = \rho. \qquad (12)\Lambda$$

That is a_1 is just the normalized covariance[5] *or* correlation coefficient.

We now prove an important inequality[6] for the coefficients a_n. Let λ be a real variable and consider the expectation

$$\langle [\theta_n^{(1)}(x_1) + \lambda\theta_n^{(2)}(x_2)]^2\rangle$$

$$= \iint_{\text{w.r.}} p(x_1\,;x_2)[\theta_n^{(1)}(x_1) + \lambda\theta_n^{(2)}(x_2)]^2\, dx_1\, dx_2 \qquad (13)$$

$$= \iint_{\text{w.r.}} p(x_1\,;x_2)[\theta_n^{(1)}(x_1)]^2\, dx_1\, dx_2$$

$$+ 2\lambda \iint_{\text{w.r.}} p(x_1\,;x_2)\theta_n^{(1)}(x_1)\theta_n^{(2)}(x_2)\, dx_1\, dx_2$$

$$+ \lambda^2 \iint_{\text{w.r.}} p(x_1\,;x_2)[\theta_n^{(2)}(x_2)]^2\, dx_1\, dx_2\,. \qquad (14)$$

On introducing the expansion (5) for $p(x_1\,;x_2)$ and carrying out the integration, making use of the orthogonal properties (2) we obtain

$$\langle [\theta_n^{(1)}(x_1) + \lambda\theta_n^{(2)}(x_2)]^2\rangle = 1 + 2\lambda a_n + \lambda^2. \qquad (15)\Lambda$$

As the expression on the left-hand side must be positive for all real λ we must have

$$a_n^2 \le 1, \qquad \text{for all } n\cdot \qquad (16)\Lambda$$

which is the required result.[7]

Application to Time Series

In the remainder of this paper we shall think of x_1 and x_2 as being sample values from a pair of time series $x_1(t)$, $x_2(t)$ at times t_1 and t_2 respectively, as it is this interpretation which has most physical significance. Thus we write

$$\left.\begin{aligned} x_1 &\equiv x_1(t_1) \\ x_2 &\equiv x_2(t_2) \end{aligned}\right\}. \qquad (17)$$

We then consider an ergodic ensemble of such pairs of time series and take $p(x_1\,;x_2)$ to be the second-order probability distribution of this ensemble. In general this probability distribution will be a function of t_1 and t_2. It then follows that the coefficients defined by (6) will be functions of t_1 and t_2 so we may write

$$a_n \equiv a_n(t_1\,;t_2). \qquad (18)\Lambda$$

In particular if the time series is stationary, only time differences are significant so that we have

$$a_n \equiv a_n(t_2 - t_1). \qquad (19)\Lambda$$

The first-order probability distributions $p_1(x_1)$ and $p_2(x_2)$, [and the corresponding sets of polynomials $\theta_n^{(1)}(x_1)$ and $\theta_n^{(2)}(x_2)$], will also, in general, be functions of t_1 and t_2, but in the stationary case will be independent of time.

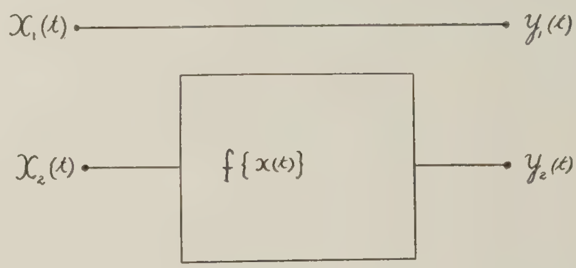

Fig. 1

Noise in "Instantaneous" Nonlinear Devices

Let us consider the system shown in Fig. 1.
The input-output relations of this system are given by

$$\left.\begin{aligned} y_1(t) &= x_1(t) \\ y_2(t) &= f\{x_2(t)\} \end{aligned}\right\}, \qquad (20)$$

where the function f is assumed not to involve differential, integral, or delay operators. In general f may be an explicit function of time, assuming that such temporal dependence is statistically independent of the input.

A device characterized by such a function f will be called[8] an "instantaneous" time-varying nonlinear device.

Let us now find the (un-normalized) covariance function for the outputs. We have

$$\Psi_{12}(t_1\,;t_2)$$

$$= \langle [y_1(t_1) - \langle y_1(t_1)\rangle][y_2(t_2) - \langle y_2(t_2)\rangle]\rangle \qquad (21)$$

$$= \langle [x_1 - \mu_1][f(x_2) - \langle f(x_2)\rangle]\rangle \qquad (22)$$

[5] Cramér, "Mathematical Methods of Statistics," Princeton University Press, Princeton, N. J., p. 265; 1946.

[6] This is just the Schwarz inequality adapted to this particular problem. See for example, Lovitt, "Linear Integral Equations," Dover Publications, New York, N. Y., p. 125; 1950.

[7] The authors are grateful to S. O. Rice for pointing out these inequalities in a private communication.

[8] The term "zero-memory" is also commonly used in place of "instantaneous."

$$= \iint_{\text{w.r.}} (x_1 - \mu_1)\{f(x_2) - \langle f(x_2)\rangle\} p(x_1 ; x_2)\, dx_1\, dx_2 \quad (23)$$

$$= \sigma_1 \iint_{\text{w.r.}} \theta_1^{(1)}(x_1)\{f(x_2) - \langle f(x_2)\rangle\} p(x_1 ; x_2)\, dx_1\, dx_2 . \quad (24)$$

To make further progress, we now expand $f(x_2)$ in a series of the polynomials $\theta_n^{(2)}(x_2)$. (We assume that the expansion can be justified.) Thus

$$f(x_2) = \sum_{m=0}^{\infty} c_m \theta_m^{(2)}(x_2) \quad (25)$$

where the coefficient $c_m \equiv c_m(t_2)$ is given by

$$c_m = \int_{\text{w.r.}} f(x_2) p_2(x_2) \theta_m^{(2)}(x_2)\, dx_2 . \quad (26)$$

In particular, remembering from (10) that $\theta_0^{(2)}(x_2) = 1$, we see that

$$c_0 = \int_{\text{w.r.}} f(x_2) p_2(x_2)\, dx_2 = \langle f(x_2)\rangle. \quad (27)$$

Thus

$$f(x_2) - \langle f(x_2)\rangle = \sum_{m=1}^{\infty} c_m \theta_m^{(2)}(x_2). \quad (28)$$

Thus using (28) in (24) we have

$$\Psi_{12}(t_1 ; t_2) = \sigma_1 \iint_{\text{w.r.}} \theta_1^{(1)}(x_1)\left\{\sum_{m=1}^{\infty} c_m \theta_m^{(2)}(x_2)\right\} p(x_1 ; x_2)\, dx_1\, dx_2 . \quad (29)$$

We now use expansion (5) for all $p(x_1 ; x_2)$ in Λ and find

$$\Psi_{12}(t_1 ; t_2) = \sigma_1 \iint_{\text{w.r.}} \theta_1^{(1)}(x_1) p_1(x_1) p_2(x_2) \cdot \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} c_m a_n \theta_n^{(1)}(x_1) \theta_m^{(2)}(x_2) \theta_n^{(2)}(x_2)\, dx_1\, dx_2 \quad (30)\Lambda$$

$$= \sigma_1 a_1 c_1 , \quad (31)\Lambda$$

where, in carrying out the integrations, use has been made of the orthogonal properties (2).

We note from (12) that a_1 is just the normalized covariance of the inputs, that is

$$a_1 = \frac{\langle (x_1 - \mu_1)(x_2 - \mu_2)\rangle}{\sigma_1 \sigma_2} = \frac{\psi_{12}(t_1 ; t_2)}{\sigma_1 \sigma_2}. \quad (32)\Lambda$$

Thus (31) may be written

$$\Psi_{12}(t_1 ; t_2) = C \cdot \psi_{12}(t_1 ; t_2), \quad (33)\Lambda$$

when

$$C \equiv C(t_2) = \int_{\text{w.r.}} f(x_2) \frac{(x_2 - \mu_2)}{\sigma_2^2}\, dx_2 . \quad (34)\Lambda$$

For stationary time series and fixed "instantaneous" nonlinear devices (33) may be written as

$$\Psi_{12}(\tau) = C \cdot \psi_{12}(\tau), \quad (35)\Lambda$$

where the constant C is independent of time and when we have written $\tau = t_2 - t_1$. The ψ and Ψ are just the ordinary cross-correlation functions of the fluctuating parts of the input and output respectively, of the system of Fig. 1. In the still more special case in which the input time series are not only stationary but identical, that is $x_1(t) = x_2(t)$, the result (35) becomes identical to that obtained by Bussgang[9] for Gaussian noise.

Luce[10] has demonstrated that Bussgang's result applies to a certain class of distributions, of which the Gaussian distribution is a special case. However Luce's distributions are of a fairly simple form[11] and it is easy to show, by means of specific examples, that there are distributions in Λ which are not of the type considered by Luce.

Some Results on Conditional Expectations

For a distribution in Λ it is easy to see from (5) that the conditional probability density for x_2 given x_1 is

$$p(x_2 \mid x_1) = \frac{p(x_1 ; x_2)}{p_1(x_1)} = p_2(x_2) \sum_{n=0}^{\infty} a_n \theta_n^{(1)}(x_1) \theta_n^{(2)}(x_2). \quad (36)\Lambda$$

Then the conditional expectation for $\theta_m^{(2)}(x_2)$ given x_1 is

$$\langle \theta_m^{(2)}(x_2) \mid x_1 \rangle = \int_{\text{w.r.}} \theta_m^{(2)}(x_2) p_2(x_2) \sum_{n=0}^{\infty} a_n \theta_n^{(1)}(x_1) \theta_n^{(2)}(x_2)\, dx_2 \quad (37)\Lambda$$

$$= a_m \theta_m^{(1)}(x_1). \quad (38)\Lambda$$

If in particular we take $m = 1$, we have, using (10) and (12)

$$\left\langle \frac{x_2 - \mu_2}{\sigma_2} \,\middle|\, x_1 \right\rangle = \rho(t_1 ; t_2) \frac{x_1 - \mu_1}{\sigma_1}. \quad (39)\Lambda$$

When $x_1(t) = x_2(t)$ is a stationary time series with normalized auto-correlation function $\rho(\tau)$, (39) becomes

$$\langle x_2 - \mu_2 \mid x_1 \rangle = \rho(\tau) \cdot (x_1 - \mu_1), \quad (40)\Lambda$$

where, as usual, we have written $\tau = t_2 - t_1$. This result has an obvious interpretation.

Stationary Markov Processes

It has been shown by Doob[12] that a stationary one-dimensional Markov process with a Gaussian probability distribution must have a correlation function which is an exponential function of time. In this section we show that this result is true for any distribution in Λ.

Let $x_1, x_2 \cdots x_n$ be samples from a stationary time series at times $t_1, t_2, \cdots t_n$.[13] If this time series is a Markov process then

$$p(x_n \mid x_1, x_2 \cdots x_{n-1}) = p(x_n \mid x_{n-1}), \quad (41)$$

by definition of a Markov process. Thus

$$\frac{p(x_1, x_2 \cdots x_n)}{p(x_1, x_2 \cdots x_{n-1})} = \frac{p(x_{n-1}, x_n)}{p(x_{n-1})} \quad (42)$$

[9] Bussgang, *loc. cit.*

[10] R. D. Luce, "Amplitude Distorted Signals," Res. Lab. Elec. Quarterly Progress Report, p. 37; April 15, 1953.

[11] A simple change of variables always reducing the second-order probability distribution to the product of independent first-order probability distributions.

[12] J. L. Doob, "Brownian motion and stochastic equations," *Ann. Math.*, vol. 43, p. 351; 1942.

[13] It is assumed that $t_n \geq t_{n-1} \geq \cdots \geq t_1$.

$$p(x_1, x_2 \cdots x_n) = p(x_1, x_2 \cdots x_{n-1}) \frac{p(x_{n-1}, x_n)}{p(x_{n-1})} \tag{43}$$

$$= \frac{p(x_1, x_2)p(x_2, x_3) \cdots p(x_{n-1}, x_n)}{p(x_2) \cdots p(x_{n-1})} \tag{44}$$

by induction.

In particular we find

$$p(x_1, x_2, x_3) = \frac{p(x_1, x_2)p(x_2; x_3)}{p(x_2)}. \tag{45}$$

If the two-dimensional distributions $p(x_1; x_2)$ is in Λ, we can use the expansion (5) in (45). We find[14]

$$p(x_1, x_2, x_3)$$
$$= p(x_1)p(x_2)p(x_3) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_m(t_2 - t_1)a_n(t_3 - t_2)$$
$$\cdot \theta_m(x_1)\theta_m(x_2)\theta_n(x_2)\theta_n(x_3). \tag{46}\Lambda$$

Let us now find the two-dimensional probability density, $p(x_1, x_3)$ by integrating out x_2 from both sides of (46).

$$p(x_1, x_3) = \int_{\text{w.r.}} p(x_1, x_2, x_3)\, dx_2 \tag{47}$$

$$= p(x_1)p(x_3) \sum_{n=0}^{\infty} a_n(t_3 - t_2)a_n(t_2 - t_1)\theta_n(x_1)\theta_n(x_3). \tag{48}\Lambda$$

But as $p(x_1, x_3)$ is an Λ we may write

$$p(x_1, x_3) = p(x_1)p(x_3) \sum_{n=0}^{\infty} a_n(t_3 - t_1)\theta_n(x_1)\theta_n(x_3). \tag{49}\Lambda$$

For (48) and (49) to be consistent we must have

$$a_n(t_3 - t_1) = a_n(t_3 - t_2)a_n(t_2 - t_1). \tag{50}\Lambda$$

It is clear that the only continuous solution of this equation is of the form

$$a_n(\tau) = e^{-\beta_n \tau}, \tag{51}\Lambda$$

where β_n is a constant depending on n.

In particular when we remember that $a_1(\tau) = \rho(\tau)$, the normalized auto correlation function, we see that

$$\rho(\tau) = e^{-\beta\tau}, \qquad \tau \geq 0, \tag{52}\Lambda$$

thus establishing the required result.

An Integral Equation Related to the Expansion (5)

The obvious similarity in form between the expansion (8) and the Mercer expansion[15] for the kernel of a linear homogeneous integral equation suggests that there may be an approach to our problem which is based on the theory of integral equations.

To demonstrate this connection we shall assume $p(x_1, x_2)$ to be a *symmetric*[16] two-dimensional probability distribution. That is, we assume

$$p(x_1, x_2) = p(x_2, x_1). \tag{53}$$

Then the expansion (5) may be written

$$p(x_1, x_2) = p(x_1)p(x_2) \sum_{n=0}^{\infty} a_n\theta_n(x_1)\theta_n(x_2). \tag{54}\Lambda$$

Multiplying both sides by $\theta_m(x_2)$ and integrating with respect to x_2 we have, using the orthogonal property (2)

$$\int_{\text{w.r.}} p(x_1, x_2)\theta_m(x_2)\, dx_2 = a_m p(x_1)\theta_m(x_1) \tag{55}\Lambda$$

which may be written

$$\int_{\text{w.r.}} \frac{p(x_1, x_2)}{\sqrt{p(x_1)p(x_2)}} \{\sqrt{p(x_2)}\ \theta_m(x_2)\}\, dx_2$$
$$= a_m\{\sqrt{p(x_1)}\ \theta_m(x_1)\} \tag{56}\Lambda$$

which is now in the standard form for a linear homogeneous integral equation

$$\int_{\text{w.r.}} K(x_1, x_2)\phi_m(x_2)\, dx_2 = \lambda_m\phi_m(x_1), \tag{57}$$

with

$$K(x_1, x_2) = \frac{p(x_1, x_2)}{\sqrt{p(x_1)p(x_2)}} \tag{58}\Lambda$$

$$\phi_m(x) = \sqrt{p(x)}\ \theta_m(x) \tag{59}\Lambda$$

$$\lambda_m = a_m. \tag{60}\Lambda$$

It follows from the definition (2) for $\theta_n(x)$ that

$$\int_{\text{w.r.}} \phi_m(x)\phi_n(x)\, dx = \int_{\text{w.r.}} p(x)\theta_m(x)\theta_n(x)\, dx = \delta_{mn} \tag{61}\Lambda$$

so that the ϕ_m defined by (59) are normalized orthogonal eigenfunctions of the linear homogeneous integral (57) with corresponding eigenvalues λ_m given by (60).

Then, providing it can be shown that the set of eigenfunctions defined by (59) is complete, the equivalence of the expansion (54) and the Mercer expansion

$$K(x_1, x_2) = \sum_{n=0}^{\infty} \lambda_n\phi_n(x_1)\phi_n(x_2) \tag{62}$$

is demonstrated.

It must be pointed out, however, that the integral equation approach does not seem to be a very suitable starting point because of the difficulties of justifying[17] the Mercer expansion and of showing when the eigenfunctions are of the form (59), which is essential for the applications discussed in this paper.

Some Examples of Distributions in Λ

In this final section we give three examples of second-order probability distributions which are in the class Λ. We shall choose distributions which are "symmetric" in the sense of (53) in order to keep the details of the analysis as simple as possible.

[14] We have written $a_n(t_2 - t_1)$, etc., in place of our more usual a_n to avoid confusion.

[15] Courant and Hilbert, *loc. cit.*, p. 138.

[16] This symmetry restriction can be removed by using theory of adjoint orthogonal functions. Courant and Hilbert, *loc. cit.*, p. 159.

[17] Sufficient, but not necessary conditions are the continuity and definiteness of the kernel $K(x_1, x_2)$. Courant and Hilbert, *loc. cit.*, p. 138.

Example 1

The second order Gaussian probability distribution has the well known form[18]

$$p(x_1, x_2) = \frac{1}{2\pi\sigma^2}(1-\rho^2)^{-1/2} \exp -\frac{1}{2}\left\{\frac{x_1^2 + x_2^2 - 2\rho x_1 x_2}{\sigma^2(1-\rho^2)}\right\} \tag{63}$$

where we have written σ, ρ for the standard deviation and normalized correlation function respectively.

The corresponding first-order probability distribution is

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\frac{x^2}{\sigma^2}\right\}. \tag{64}$$

We now make use of Mehler's expansion,[19] namely

$$(1-p^2)^{-1/2} \exp -\frac{1}{2}\left\{\frac{p^2(x_1^2+x_2^2) - 2px_1x_2}{(1-p^2)}\right\} = \sum_{n=0}^{\infty} p^n \frac{H_n(x_1)H_n(x_2)}{n!}, \tag{65}$$

where $H_n(x)$ is the Hermite Polynomial of degree n defined[20] by

$$\frac{d^n}{dx^n}\{e^{-x^2/2}\} = (-1)^n H_n(x) e^{-x^2/2}. \tag{66}$$

Then using (65) it follows that (63) may be written

$$p(x_1, x_2) = \frac{1}{2\pi\sigma^2} \exp -\frac{1}{2}\left\{\frac{x_1^2 + x_2^2}{\sigma^2}\right\} \cdot \sum_{n=0}^{\infty} \rho^n \frac{H_n(x_1/\sigma)H_n(x_2/\sigma)}{n!}. \tag{67}$$

The Hermite Polynomials have the orthogonal property[21]

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-x^2/2} H_m(x)H_n(x)\,dx = \delta_{mn} n! \tag{68}$$

so it is seen that (67) can finally be written in the standard form[22]

$$p(x_1, x_2) = p(x_1)p(x_2)\sum_{n=0}^{\infty} a_n \theta_n(x_1)\theta_n(x_2) \tag{69Λ}$$

with

$$\left.\begin{aligned} \theta_n(x) &= (n!)^{-1/2} H_n(x) \\ a_n &= p^n \end{aligned}\right\} \tag{70}$$

showing that $p(x_1, x_2)$ is in Λ.

Example 2

When Gaussian noise is applied to a narrow bandpass filter, the output of the filter behaves as a sine wave at the midband frequency with random, relatively slowly varying amplitude and phase. Rice has shown[23] that the second-order probability distribution for the envelope R of the filter output is

$$p(R_1, R_2) = \frac{R_1R_2}{\sigma^4}(1-\mu^2)^{-1} I_0\left\{\frac{R_1R_2}{\sigma^2}\frac{\mu}{1-\mu^2}\right\} \cdot \exp -\frac{1}{2}\left\{\frac{R_1^2 + R_2^2}{\sigma^2(1-\mu^2)}\right\}. \tag{71}$$

Here $I_0(x)$ is the modified Bessel Function of zero order and

$$\mu^2 \equiv \mu^2(\tau) = \frac{\int_0^\infty \int_0^\infty w(f_1)w(f_2)\cos 2\pi(f_1 - f_2)\tau\, df_1\, df_2}{\left\{\int_0^\infty w(f)\,df\right\}^2} \tag{72}$$

$$\sigma^2 = \int_0^\infty w(f)\,df, \tag{73}$$

when $w(f)$ is the power spectrum of the noise at the output of the filter.

The corresponding first order distribution has the very simple form (Rayleigh distribution)

$$p(R) = \frac{R}{\sigma^2}\exp\left\{-\frac{1}{2}\frac{R^2}{\sigma^2}\right\}. \tag{74}$$

We now suppose that an "instantaneous" square law envelope detector is connected to the output of our narrow band filter. Such a detector would have a time constant, long compared to the reciprocal of the filter midband frequency, but short compared to the reciprocal of the filter bandwidth. These are conflicting requirements, but are usually closely approximated in practice. Then denoting the outputs at times t_1, t_2 by x_1 and x_2 respectively, we may write

$$\left.\begin{aligned} x_1 &= R_1^2 \\ x_2 &= R_2^2 \end{aligned}\right\}. \tag{75}$$

From (71), we find immediately

$$p(x_1, x_2) = \frac{(1-\mu^2)^{-1}}{4\sigma^4} I_0\left\{\frac{\sqrt{x_1x_2}\,\mu}{\sigma^2 1-\mu^2}\right\} \exp -\frac{1}{2}\left\{\frac{x_1 + x_2}{\sigma^2(1-\mu^2)}\right\} \tag{76}$$

and from (74), we have for the first-order distribution

$$p(x) = \frac{1}{2\sigma^2}\exp\left\{-\frac{1}{2}\frac{x}{\sigma^2}\right\}. \tag{77}$$

We now make use of the identity[24]

$$(1-t)^{-1}\exp\left\{-(x+y)\frac{t}{1-t}\right\} I_0\left\{\frac{2\sqrt{xyt}}{1-t}\right\} = \sum_{n=0}^{\infty} t^n L_n(x)L_n(y), \tag{78}$$

when $L_n(x)$ is the Laguerre Polynomial defined[25] by

$$\frac{d^n}{dx^n}\{x^n e^{-x}\} = n!\, e^{-x} L_n(x). \tag{79}$$

[18] S. O. Rice, *Bell Sys. Tech. Jour.*, vol. 23, p. 50; 1945. We have assumed zero means in this example for simplicity.

[19] G. N. Watson, "Generating functions for polynomials II," *Jour. London Math. Soc.*, vol. 8, p. 194; 1933.

[20] Cramér, *loc. cit.*, p. 133.

[21] Cramér, *loc. cit.*, p. 133.

[22] This expansion is a fairly well-known one (e.g. Cramér, p. 133) and has been used by Siegert in noise problems; see "On the evaluation of noise samples," *Jour. Appl. Phys.*, vol. 23, no. 7; 1952.

[23] S. O. Rice, *Bell Sys. Tech. Jour.*, vol. 23, p. 78; 1945.

[24] G. N. Watson, "Generating functions of polynomials I," *Jour. London Math. Soc.*, vol. 8, p. 189; 1933. This is a special case of the more general result given by Watson.

[25] G. Szego, "Orthogonal polynomials," *Amer. Math. Soc. Colloquium Pub.*, vol. 23, p. 97; 1939.

Using (78) we may write (76) in the form

$$p(x_1, x_2) = \frac{1}{4\sigma^4} \exp -\frac{1}{2}\left\{\frac{x_1 + x_2}{\sigma^2}\right\} \sum_{n=0}^{\infty} (\mu^2)^n L_n\left(\frac{x_1}{2\sigma^2}\right) L_n\left(\frac{x_2}{2\sigma^2}\right) \tag{80}$$

$$= p(x_1)p(x_2) \sum_{n=0}^{\infty} (\mu^2)^n L_n\left(\frac{x_1}{2\sigma^2}\right) L_n\left(\frac{x_2}{2\sigma^2}\right) \tag{81}$$

where use has been made of (77). When it is recalled that the orthogonal property for the Laguerre Polynomials is[26]

$$\int_0^{\infty} e^{-x} L_m(x) L_n(x)\, dx = \delta_{mn} \tag{82}$$

it is seen that (81) is already in standard form with

$$\left.\begin{aligned} \theta_n(x) &= L_n\left(\frac{x}{2\sigma^2}\right) \\ a_n &= (\mu^2)^n \end{aligned}\right\} \tag{83}$$

showing that $p(x_1, x_2)$ defined by (75) is in Λ.

Example 3

As a final example, we show that the second-order probability distribution for a sine wave of constant amplitude P is in Λ. It will be most convenient in this example to start from the characteristic function of the distribution. If we denote our sine wave by

$$x(t) = p \cos(wt + \phi), \tag{84}$$

the second-order characteristic function defined as

$$g(u, v) = \langle e^{iux_1 + ivx_2} \rangle \tag{85}$$

has been shown by Rice[27] to be

$$g(u, v) = J_0\{p\sqrt{u^2 + v^2 + 2uv \cos w\tau}\}. \tag{86}$$

It follows that the second order probability distribution $p(x_1, x_2)$ is given by the Fourier transform,

$$p(x_1, x_2) = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} J_0\{p\sqrt{u^2 + v^2 + 2uv \cos w\tau}\} \cdot e^{-iux_1 - iux_2}\, du\, dv. \tag{87}$$

To make further progress, we use Neumann's addition theorem,[28] namely

$$J_0\{\sqrt{u^2 + v^2 - 2uv \cos \phi}\} = \sum_{n=0}^{\infty} \epsilon_n J_n(u) J_n(v) \cos n\phi, \tag{88}$$

where

$$\left.\begin{aligned} \epsilon_n &= 1 \qquad n = 0 \\ &= 2 \qquad n = 1, 2, \cdots \end{aligned}\right\}. \tag{89}$$

Using (88) in (87) and reversing the order of summation and integration, we find

$$p(x_1, x_2) = \sum_{n=0}^{\infty} (-1)^n \epsilon_n \cos nw\tau \cdot \left\{\frac{1}{2\pi}\int_{-\infty}^{+\infty} J_n(pu) e^{-iux_1}\, du\right\}\left\{\frac{1}{2\pi}\int_{-\infty}^{+\infty} J_n(pv) e^{-ivx_2}\, dv\right\}. \tag{90}$$

We now use the result[29]

$$\left.\begin{aligned} \frac{1}{2\pi}\int_{-\infty}^{+\infty} J_n(pu) e^{-iux}\, du &= \frac{(-i)^n T_n(x/p)}{\pi(p^2 - x^2)^{1/2}}, \quad \frac{x^2}{p^2} < 1 \\ &= 0, \qquad\qquad \frac{x^2}{p^2} > 1 \end{aligned}\right\}. \tag{91}$$

Here $T_n(x)$ is the Tchebycheff Polynomial of the first kind defined by

$$T_n(x) = \cos\{n \cos^{-1} x\}. \tag{92}$$

Using (91), we see that (90) may be written

$$p(x_1, x_2) = \frac{1}{\pi^2}(p^2 - x_1^2)^{-1/2}(p^2 - x_2^2)^{-1/2} \cdot \sum_{n=0}^{\infty} \epsilon_n T_n\left(\frac{x_1}{p}\right) T_n\left(\frac{x_2}{p}\right) \cos nw\tau, \tag{93}$$

when

$$\frac{x_1^2}{p^2} < 0, \qquad \frac{x_2^2}{p^2} < 0$$

otherwise

$$p(x_1, x_2) = 0.$$

The first-order probability distribution for the sine wave has the well-known form[30]

$$p(x) = \frac{1}{\pi}(p^2 - x^2)^{-1/2} \tag{94}$$

and the Tchebycheff Polynomial has the orthogonal property

$$\frac{1}{\pi}\int_{-1}^{+1} \epsilon_n T_m(x) T_n(x) (1 - x^2)^{-1/2}\, dx = \delta_{mn} \tag{95}$$

so it follows that (93) may be written in the standard form

$$p(x_1, x_2) = p(x_1)p(x_2) \sum_{n=0}^{\infty} a_n \theta_n(x_1)\theta_n(x_2), \tag{96}$$

where

$$\left.\begin{aligned} \theta_n(x) &= \sqrt{\epsilon_n}\, T_n\left(\frac{x}{p}\right) \\ a_n &= \cos nw\tau \end{aligned}\right\}. \tag{97}$$

Note that, from (92), we may write $a_n = T_n(a_1)$.

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[26] *Ibid.*

[27] S. O. Rice, "Mathematical analysis of random noise," *Bell Sys. Tech. Jour.*, vol. 24, p. 138; 1945.

[28] G. N. Watson, "Theory of Bessel Functions," Cambridge University Press, Cambridge, Eng., p. 358; 1922.

[29] P. R. Clement, "The Chebyshev Approximation Method," *Quart. Appl. Math.*, vol. 11; July, 1953.

[30] Rice, "Mathematical analysis of random noise," *op. cit.*, p. 48.

Predictive Coding—Part I

PETER ELIAS†

Summary—**Predictive coding is a procedure for transmitting messages which are sequences of magnitudes. In this coding method, the transmitter and the receiver store past message terms, and from them estimate the value of the next message term. The transmitter transmits, not the message term, but the difference between it and its predicted value. At the receiver this error term is added to the receiver prediction to reproduce the message term. This procedure is defined and messages, prediction, entropy, and ideal coding are discussed to provide a basis for Part II, which will give the mathematical criterion for the best predictor for use in the predictive coding of particular messages, will give examples of such messages, and will show that the error term which is transmitted in predictive coding may always be coded efficiently.**

Introduction

Two major contributions have been made within the past few years to the mathematical theory of communication. One of these is Wiener's work on the prediction and filtering of random, stationary time series, and the other is Shannon's work, defining the information content of a message which is such a time series, and relating this quantity to the bandwidth and time required for the transmission of the message.[1] This paper makes use of the point of view suggested by Wiener's work on prediction to attack a problem in Shannon's field: prediction is used to make possible the efficient coding of a class of messages of considerable physical interest.

Consider a message which is a time series, a function m_i which is defined for all integer i, positive or negative. Such a series might be derived from the sampling used in a pulse-code modulation system.[2] From a knowledge of the statistics of the set of messages to be transmitted, we may find a predictor which operates on all the past values of the function, m_j with j less than i, and produces a prediction p_i of the value which m will next assume. Now consider the error e_i, which is defined as the difference between the message and its predicted value:

$$e_i = m_i - p_i . \tag{1}$$

All of the information generated by the source in selecting the term m_i is given just as well by e_i; the error term may be transmitted, and will enable the receiver to reconstruct the original message, for the portion of the message that is not transmitted, p_i, may be considered as information about the *past* of the message and not about its present; indeed, since p_i is a quite determinate mathematical function, it contains no information at all by Shannon's definition of this quantity.[3]

The communications procedure which will be discussed is illustrated in Fig. 1. There is a message-generating source that feeds into a memory at the transmitter. The transmitter has a predictor, which operates on the past of the message as stored in the memory to produce an estimate of its future. The subtractor subtracts the prediction from the message term and produces an error term e_i, which is applied as an input to the coder. The coder codes the error term, and this coded term is sent to the receiver. In the receiver the transmitting process is reversed. The receiver also has a memory and an identical predictor, and has predicted the same value p_i for the message as did the predictor at the transmitter. When the coded correction term is received, it is decoded to reproduce the error term e_i. This is added to the predicted value p_i and the message term m_i is reproduced. The message term is then presented to the observer at the receiver, and is also stored in the receiver memory to permit the prediction of the following values of the message.

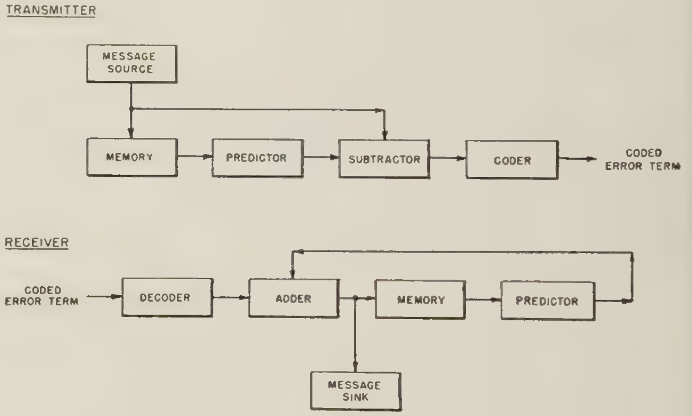

Fig. 1—Predicting coding and decoding procedure.

This procedure is essentially a coding scheme, and will be called *predictive coding*. The memory, predictor, subtractor, and coder at the transmitter, and the memory, predictor, adder, and decoder at the receiver may be considered as complex coding and decoding devices. Predictive coding may then be compared with the ideal coding methods given by Shannon and Fano.[4] In general,

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[1] For historical remarks on the origin of modern information theory see C. E. Shannon and W. Weaver, "The Mathematical Theory of Communication," Univ. of Illinois Press, Urbana, Ill., p. 52 (footnote) and p. 95 (footnote); 1949.

[2] B. M. Oliver, J. R. Pierce, and C. E. Shannon, "The philosophy of PCM," Proc. I.R.E., vol. 36, pp. 1324–1331; November, 1948; also, W. R. Bennett, "Spectra of quantized signals," *Bell Sys. Tech. Jour.*, vol. 27, pp. 446–472; July, 1948.

[3] Shannon and Weaver, *op. cit.*, p. 31.

[4] Shannon and Weaver, *op. cit.*, p. 30; also R. M. Fano, Tech. Rep. No. 65, Res. Lab. Elect., M.I.T., Cambridge, Mass.; 1949.

predictive coding cannot take less channel space for the transmission of a message at a given rate than does an ideal coding scheme, and it will often take more. However, there is a large class of message-generating processes which are at present coded in a highly inefficient way, and for which the use of large codebook memories, such as are required for the ideal coding methods, is impractical. Time series which are obtained by sampling a smoothly varying function of time are examples in this class. For many such processes predictive coding can give an efficient code, using a reasonable amount of apparatus at the transmitter and the receiver.

It should be noted that in the transmission scheme of Fig. 1 errors accumulate. That is, any noise which is introduced after the transmitter memory, or at the receiver, or in transmission, will be perpetuated as an error in all future values of the message, as will any discrepancy between the operation of the two memories, or the two predictors. This means that eventually errors will accumulate to such an extent that the message will disappear in the noise. If, therefore, continuous messages, i.e., time series each member of which is selected from a continuum of magnitudes, are to be transmitted, it will be necessary periodically to clear the memories of both the receiver and the transmitter and start afresh. This is undesirable, since after each such clearing there will be no remembered values on which to base a prediction, and more information transmission will be required for a period following each such clearing, until enough remembered values have accumulated to permit good prediction once more.

A more satisfactory alternative is the use of some pulse-code transmission system in which only quantized magnitudes of input are accepted. Such a system may be made virtually error-free.[5] A system of this kind has the further advantage that the only very reliable memory units now available or in immediate prospect are of a quantized nature, most of them being capable only of storing binary digits. The use of a quantized system requires that the predicted values be selected from the permissible quantized set of message values. Strictly interpreted, this severely limits the permissible predictors; if by a choice of scale the permissible quantized levels are made equal to the integers, then the restriction on $p(m_{i-1} \cdots m_{i-n})$ is that it take integer values for all sets of integer arguments. Actually the ordinary extrapolation formulas have this property, and may be used as predictors. But it is not necessary to limit the choice of predictors so severely. The problem may be evaded by using any function as a predictor and computing its value to a predetermined number of places by digital computing techniques, the prediction then being taken to be the function rounded off to the nearest integer. If the predictor as originally computed was optimum in some well-defined sense, then the rounded predictor will presumably be less good in that sense, but in cases where predictive coding may be expected to be useful the difference will usually be small.

[5] Oliver, Pierce, and Shannon, *loc. cit.*

It is necessary to define precisely what is meant by an optimum predictor for use in predictive coding—i.e., to define some quantity, which depends upon the choice of the predictor, and define as optimum a predictor which minimizes this quantity. Wiener's work uses as a criterion the minimization of the mean square error term $\overline{e^2}$. Wiener has pointed out that other criteria are possible, but that the mathematical work is made simpler by the mean square choice.[6] Minimizing the mean square error corresponds to minimizing the power of the error term, and if no further coding is to be done, this is a reasonable criterion for predictive coding purposes. However, in the system illustrated in Fig. 1, the error term is coded before it is transmitted, and its power may be radically altered in the coding process. What we are really interested in minimizing is the channel space which the system will require for the transmission of the error term. This leads to the following criterion which will be justified in Part II of this paper: *That predictor is best which leads to an average error-term distribution having minimum entropy.*

The coder of Fig. 1 also requires some consideration. Predictive coding eliminates the codebook requirement by using prediction. To take advantage of the resultant savings in equipment, it is necessary to show that the coder itself will not require a large codebook. This reduces to the problem of showing that a message whose terms are assumed independent of one another may always be coded efficiently by a process with a small memory requirement. It will be shown that this is true. It is necessary to use two kinds of coding processes: one for cases in which the entropy of the distribution from which the successive terms are chosen is large compared to unity, and another for cases in which the entropy is small compared to unity.

The following sections of the present paper are devoted to a discussion of messages, prediction, entropy, and ideal coding. Part II will discuss the predictor criterion given above, the classes of messages for which a predictor that is optimum by this criterion may be found, and other classes of messages for which predictive coding may be of use. Mathematically defined examples of message-generating processes which belong to these classes will be given, and the problem of coding the error term so as to take advantage of the minimal entropy of its average distribution will be examined.

Characterization of Messages

A necessary preliminary to a discussion of messages is a precise definition of what "message" is taken to mean.[7] Since a communication system is designed to transmit many messages, what is actually of interest is the

[6] N. Wiener, "The Extrapolation, Interpolation and Smoothing of Stationary Time Series with Engineering Applications," published in 1942 as an NDRC report, and in 1949 as a book, by the Mass. Inst. Tech. Press, Cambridge, Mass., and John Wiley & Sons, Inc., New York, N. Y., especially p. 13.

[7] Such definitions are given by Wiener, *ibid.*, and Wiener, "Cybernetics," Mass. Inst. Tech. Press, and John Wiley & Sons, Inc., 1948; also by Shannon and Weaver, *loc. cit.* Our discussion starts with a definition like Wiener's and ends with one like Shannon's.

characterization of the ensemble from which the transmitted messages are chosen, or the stochastic process by which they are generated. As a preliminary definition, we may say that a message is a single-valued real function of time, chosen from an ensemble of such functions. It will be denoted by $m(a, t)$, where a is a real number between zero and one which labels the particular message chosen from the ensemble, and $m(a, t)$ is defined, for each such a, for all values of t from $-\infty$ to ∞. This definition must be restricted in several respects, in part to take into account the physical requirements of transmitting systems and in part for mathematical convenience.

First, it is assumed that the ensemble from which the messages are chosen is ergodic. This means that any one message of the ensemble, except for a set whose measure in a is zero, is typical of the ensemble in the following sense: let $Q(a)$ be the probability distribution of the parameter of distribution a. Then with probability one, for any function $f[m(a, t)]$ and almost any a_1,

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f[m(a_1, t)]\, dt = \int_0^1 f[m(a, t)]\, dQ(a). \qquad (2)$$

I.e., any function of m has the same average value when averaged over time as a function of a single message, as when averaged over the ensemble of all possible messages. We can thus find out all possible statistical information about the ensemble by observing a single message over its entire history. The ergodic requirement implies that the ensemble is stationary: i.e., that the statistics do not change with time. Its practical importance is that it permits us to speak indifferently of the message or the ensemble, and makes it unnecessary to specify the sense in which we speak of an average. In particular, it permits the substitution of measurable time averages for experimentally awkward ensemble averages.

Second, it is assumed that the average square of the message [in either sense of (2)] is finite. The message will be represented in physical systems by a voltage or a current, or the displacement of a membrane, or the pressure in a gas, or by several such physical variables, as it proceeds from its origin to its destination. All of these representations require power; in particular, representation as a voltage or a current between two points separated by a fixed impedance, which is a necessary intermediate representation in any presently used electrical communication method, requires a power proportional to the square of the message. Since only a finite amount of power may be supplied to a physical transmitter, it is obviously required that the average message power be bounded.

Third, it is assumed that the spectrum of the message vanishes for frequencies greater than some fixed frequency f_0. This will not in general be true for the radio-frequency spectrum of the messages as they are generated by a source, and it has been shown that a function with an infinitely extended spectrum cannot be reduced to a function with a spectrum of finite range by any physically realizable filter; the transfer characteristic of a filter can be zero only for a set of frequencies of total measure zero.[8] However, this is no practical problem. For since the message has a finite total power distributed over the spectrum, there will always be an f_0 so high that a negligible fraction of the total power will be located beyond it in the power spectrum.

The reason for this assumption is that, as Shannon has pointed out, any function of time that is band-limited may be replaced by a time series, which gives the values of the function at times separated by an interval $1/2f_0$.[9] For any band-limited function we have the following identity:

$$m(t) = \sum_{i=-\infty}^{\infty} m(i/2f_0)\left\{\frac{\sin \pi(2f_0 t - i)}{\pi(2f_0 t - i)}\right\}. \qquad (3)$$

The values of the function at the sampling points $t = i/2f_0$, which are the coefficients of this series, thus completely determine the function. If the function is not initially band-limited, the expansion will give a function which passes through the same values at the sampling points, but which is band-limited. As we assume band-limited messages, for our purpose the series and the function are equivalent, and since the series is easier to deal with in the sequel, it is desirable to change the definition of the message. Henceforth the message will be defined as the series of coefficients in the expansion (3). By choice of the unit of time, the sampling interval is made unity, and the message is then $m_i(a)$, defined for all (positive and negative) integer values of the index i.

A message is thus a time series drawn from an ergodic ensemble of such series, and each term in any one message is drawn from a probability distribution whose form is determined by the preceding terms of that message. For the reasons indicated in the first section, we will be interested primarily in quantized messages, for which this probability distribution will be discrete. However, it will at times be more convenient in the analysis and the examples to deal with continuous distributions, it being understood that quantization will ultimately be used. In the discrete case, the message term m_i will be selected from a discrete probability distribution M_k, where $M_k(m_{i-1} \cdots m_{i-j} \cdots)$ is the conditional distribution giving the probability that, for a particular set of past values $m_{i-1} \cdots m_{i-j} \cdots$, the message term m_i will take the integer value k. In the continuous case, the message term m_i will be chosen from a continuous conditional distribution $M(m_i : m_{i-1} \cdots m_{i-j} \cdots)$. Both of these distributions are dependent on the set of values of the preceding message terms $m_{i-1} \cdots m_{i-j} \cdots$, but are of course independent of the value of the index i, by the stationary nature of the ensemble.

[8] Wiener, "The Extrapolation, Interpolation and Smoothing of Stationary Time Series with Engineering Applications," NDRC Report, Mass. Inst. Tech. Press, Cambridge, Mass., p. 37; 1942.

[9] C. E. Shannon, "Communication in the presence of noise," Proc. I.R.E., vol. 37, pp. 10–21; January, 1949.

Stochastic processes of this sort are known as Markoff processes and have an extensive mathematical literature.[10] An nth order Markoff process is one in which the distribution from which each term is chosen depends on the set of values of the n preceding terms only; a process in which each term is chosen from a single unconditional probability distribution may be called a Markoff process of order zero. It should be noted that, while any Markoff process yielding a message with a finite second moment is included in this definition, we will expect most of the messages to be Markoff processes of a rather special kind. The messages have been derived by the time-sampling of a continuously varying physical quantity. The sampling rate must be high enough so that the sampling does not suppress significant variations in the message—i.e., the f_0 must be above the bulk of the spectral power of the message. Now for most such messages, the average rate of variation with time is much lower than the highest rate that the system must be capable of transmitting. Consequently, it is to be expected that on the average, successive message values will be near to one another. This means, in particular, that in the discrete case the index k is not just an arbitrary labeling of a particular symbol—as it is, for example, in Shannon's finite-order Markoff approximations to English[11]—but may be expected to give a genuine metric: message values with indexes near to one another may be expected to have probabilities near to one another, and the conditional distributions mentioned above may be expected to be unimodal. This is not a restriction on what kinds of series will be considered to be messages, but is rather a specification of the class of messages for which predictive coding may be expected to be of use, as will be discussed in detail in Part II of this paper.

For a message ensemble for which the conditional distributions are not given *a priori*, it is necessary to determine them by the observation of a number of messages, or of a single message for a long time. It is obviously impossible to do this on the assumption that the distribution from which a particular message term is chosen depends on the infinite set of past message values. What can, in fact, be measured are the zeroth order approximation, in which each term is treated as if it were drawn from the same distribution, giving $M(m_i)$, an unconditional distribution; the first order conditional distribution $M(m_i : m_{i-1})$, and so on to the nth order conditional distribution for some finite n. A communications system which is designed to transmit this approximation will be inefficient: the approximating process itself would generate messages with a greater information content than the messages which are actually being transmitted, and a system designed for the approximation will waste time or power or bandwidth when transmitting the real message. This will be discussed more fully later.

[10] Shannon and Weaver, *op. cit.*, p. 15; also, M. Frechet, cited there, and P. Levy, "Processus Stochastique et Mouvement Brownien," Gauthier-Villars, 1948, which give further references.

[11] Shannon and Weaver, *op. cit.*, pp. 9–15.

Prediction

Norbert Wiener has developed a very general method for finding the linear predictor for a given ensemble of messages which minimizes the root mean square error of prediction. His method was developed for the difficult case of nonband-limited messages, i.e., continuous functions of time which cannot be reduced to time series. However, he has also solved the much simpler problem of the prediction of time series, such as the messages which were defined above. The details of this work are thoroughly covered in the literature,[12] and this section will merely define some terms, note some results, and discuss the prediction problem from a point of view which is weighted towards probability considerations and not towards Fourier transform considerations.

From a time series, a linear prediction p_i of the value of a message term m_i is a linear combination of the previous message values

$$p_i = \sum_{j=1}^{\infty} a_j m_{i-j} .$$

The error e, is defined as

$$e_i = p_i - m_i .$$

The predictor itself may be considered to be the set of coefficients a_j. The best linear predictor, in the rms sense, is the set of coefficients which, on the average, minimizes e^2. Wiener has shown that this predictor is determined, not by the message ensemble directly, but by the autocorrelation function of the ensemble. In general, there will be many ensembles with the same autocorrelation function, and the same linear predictor will be the best in the rms sense for all of them.

The autocorrelation function for a time series is defined by

$$c_k = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{i=-N}^{N} m_i m_{i-k} .$$

Devices for rapidly obtaining approximate autocorrelation functions have been constructed.[13] These devices accept the message directly as an input, and graph or tabulate the function. By the use of such devices, or by a statistical examination of the message, or in some cases by an *a priori* knowledge of the message-generating process, it is possible to determine the autocorrelation function. The best linear predictor in the rms sense may then be determined. But it should be noted that there may be nonlinear predictors which are very much better.

Indeed, given a complete knowledge of the stochastic definition of the message, i.e., a complete knowledge of

[12] Wiener, *op. cit.* Also H. W. Bode and C. E. Shannon, "A simplified derivation of linear least square smoothing and prediction theory," Proc. IRE, vol. 38, pp. 417–425; April, 1950.

[13] T. P. Cheatham, Jr., Tech. Rep. No. 122, Res. Lab. Elect., M. I. T. (to be published). See also, Y. W. Lee, T. P. Cheatham, Jr., and J. B. Wiesner, "The Application of Correlation Functions in the Detection of Small Signals in Noise," Tech. Rep. No. 141, Res. Lab. Elect., M. I. T.; 1949.

the conditional probability distributions $M(m_i : m_{i-1} \cdots m_{i-j} \cdots)$ or $M_k(m_{i-1} \cdots m_{i-j} \cdots)$ the *best* rms predictor, with no restriction as to linearity, is directly available. Obviously the best rms predictor for a message term m_i defined in this way is the mean of the distribution from which it is chosen, which is determined by the past message history: i.e., the best rms predictor, p^*, is

$$p^* = \bar{m}_i = \sum_{k=-\infty}^{\infty} k M_k(m_{i-1} \cdots m_{i-j} \cdots)$$

or

$$= \int_{-\infty}^{\infty} m_i M(m_i : m_{i-1} \cdots m_{i-j} \cdots)\, dm_i$$

in the discrete and continuous cases respectively. For the mean of a distribution is that point about which its second moment is a minimum. Of course, the mean need not be a linear function of the past message values. However, it is some determinate function of these values unless the message values are completely uncorrelated—i.e., unless the Markoff process is of order zero. In this case, it is just the constant which is the mean of the zero-order distribution. We therefore have as the unconditionally best rms predictor the function $p^*(m_{i-1} \cdots m_{i-j} \cdots)$.

From this same general statistical viewpoint the best predictor on a mean-absolute error basis is the prediction of the median of the conditional distribution, since the median is that point about which the first absolute moment is a minimum. Like the mean, the median is defined by the conditional distribution M as a function of the past history of the message. This definition may not be unique: if there is a region of zero probability density between the two halves of a probability distribution, any point in the region is a median. However, the definition may be made unique by selecting a point within this range, for those sets of past message values for which the ambiguity arises. We will denote the best predictor in the mean-absolute sense by p^{**}, it being understood that the definition has been made unique in some suitable way if the ensemble is such as to require this.

Finally, it may be desired to predict in such a way that in the discrete case, the probability of no error is a maximum, and in the continuous case the probability density has the maximum possible value at zero error. This requires modal prediction. The mode of the conditional distribution will not be unique if there are several equal probabilities which are each larger than any other probability in the discrete case, or if the continuous distribution attains its maximum value at more than one point. The difficulty may again be removed by a suitable choice, and p^{***} will signify the best modal predictor.

In any of these cases, and indeed for any other prediction criterion which yields a determinate value of the prediction as a function of the past history of the message, the error term e_i is drawn from a distribution $E(e_i : m_{i-1} \cdots m_{i-j} \cdots)$ or $E_k(m_{i-1} \cdots m_{i-j} \cdots)$ which is of exactly the same form as the original distribution of the message term, but which has been shifted along the axis by the amount of the prediction. If it is desired to limit predictions to the possible quantized values of a discrete probability distribution, it is only necessary to make p^{**} and p^{***} unique in a way which does this in the cases of ambiguity; where the median and mode are uniquely defined, they will always coincide with one of the possible values of the message. For rms prediction it is necessary to take the quantized value that is nearest to the computed mean of the distribution as the value of p^*.

As an example of a predictable function, consider

$$M(m_i : m_{i-1}) = \frac{1}{\sigma\sqrt{2\pi}} \exp\,[-(m_i - am_{i-1})^2/2\sigma^2]. \quad (4)$$

The unconditional distribution of m_i may be found by using the reproductive property of the normal distribution. $\bar{M}(m_i)$ will be normal, with a standard deviation σ', and am_{i-1} will have a normal distribution with standard deviation $a\sigma'$: then,

$$\sigma^2 + a^2\sigma'^2 = \sigma'^2; \qquad \sigma' = \frac{\sigma}{\sqrt{1 - a^2}} \quad (5)$$

and

$$\bar{M}(m_i) = \frac{1}{\sigma'\sqrt{2\pi}} \exp\,[-m_i^2/2\sigma'^2]. \quad (6)$$

The zero-order approximation to this first-order Markoff process has, then, a message term distribution of the same form as the original conditional distribution, but a standard deviation which is larger by a factor $1/\sqrt{1 - a^2}$. By our definition in a previous section the process will generate messages only if $a < 1$: otherwise the standard deviation will be infinite, and the message will require infinite power for transmission. A more general example in complete analogy to (4) is:

$$M(m_i : m_{i-1} \cdots m_{i-j} \cdots)$$
$$= \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\left(m_i - \sum_{j=1}^{\infty} m_{i-j}a_j\right)^2 \Big/ 2^2\right]. \quad (7)$$

Wiener's prediction procedure is designed for functions of the form (7), in which each term of the time series is drawn from a normal distribution with constant σ, with a mean which is a linear combination of past values, the permissible combinations being limited by the requirement that the resultant average distribution have a finite second moment. The linear combination of past values which is the mean of the conditional distribution is also the best linear rms predictor, and is indeed *the* best rms predictor p^*, as noted above. Wiener's method is then a procedure for finding this linear combination in terms of the autocorrelation function of the message.

The combination of past terms in the exponent may be rewritten as a sum of differences, less a constant times the message value m_i. The stochastic function determined by the conditional distribution will then be as approximation to the solution of the difference equation obtained by setting the exponent in (7) equal to zero. In the limit $\sigma \to 0$, the stochastic function will become precisely the

function which is a solution to this equation, as determined by the set of past message values (initial conditions): as σ grows, the function will wander about in the neighborhood of this solution, diverging from it more and more as i increases. In (4) above, the equation obtained is just $m_i - am_{i-1} = 0$, and the solution, $m_i = am_{i-1}$, gives a geometric approach to the origin.

In the case of continuous functions of time, taking appropriate limits gives a normal distribution about a linear function of the past which may include integral or differential operators on the past. The bulk of Wiener's analysis is devoted to this case. Although the method was designed with functions like (7) in mind, it is clearly not limited to them. In the case of time series it is possible to use a distribution which is not normal, with a standard deviation (or other parameter or parameters) which is not constant, but is also determined by the past values of the message. So long as the *mean* of the distribution is still a linear combination of past values, the predictor derived from the autocorrelation function will still give the best rms predictor. If the mean is a nonlinear function of the past values, the predictor obtained from the autocorrelation function will be the best linear approximation to this nonlinear function in the rms sense.

Where the best predictor is indeed linear, or is well approximated by a linear combination of past values, the great practical superiority of Wiener's method over the use of the conditional distribution should be clear. For in this method only the autocorrelation function, a function of a single variable, needs to be measured; the predictor can then be computed no matter what the order of the Markoff process may be. Using the conditional probability distribution directly, an nth order Markoff process will require the observational determination of a function of $n + 1$ variables. This becomes a task of fantastic proportions when n is as large as four or five: it is practical for small n only for a quantized system with very few possible quantized levels.

The direct use of the conditional distribution may, however, be quite valuable if the best rms predictor is a highly nonlinear function of only a few past values, particularly in a quantized system. Nonlinearity is no more difficult to treat than is the linear case as far as analysis by this method is concerned. For the synthesis problem the lack of suitable nonlinear elements for the physical construction of nonlinear operators on the past is confined to the case of continuous functions of time; in the case of time series with quantized terms, digital computer techniques can provide any desired nonlinear function of any number of variables—at, of course, an expense in equipment which may become very large for large n.

When the conditional distribution always has a point of symmetry, we may note that the best rms predictor p^* is equal to the best mean absolute predictor p^{**}. If the distribution is also always unimodal, then the best modal predictor p^{***} will also be the same as p^*. In particular, this will be the case for the examples (4) and (7), but it does not, of course, depend on the linearity of the predictor.

Entropy, Averaging, and Ideal Coding

The entropy H of a probability distribution M has been defined as[14]

$$H = -\sum_{k=-\infty}^{\infty} M_k \log M_k$$

and

$$H = -\int_{-\infty}^{\infty} M(m_i) \log M(m_i) \, dm_i \tag{8}$$

in the discrete and continuous cases, respectively. The entropy of a probability distribution may be used as a measure of the information content of a symbol or message value m_i chosen from this distribution. The choice of the logarithmic base corresponds to the choice of a unit of entropy: when logarithms are taken to the base two, as is convenient in many discrete cases, the unit of entropy is the "bit," a contraction for binary digit, since in a two-symbol system with the two symbols equiprobable, the entropy per symbol is one bit for this choice of base. In the continuous case computations are often made simpler by the use of natural logarithms. The resultant unit of entropy is called by Shannon the natural unit. We have one natural unit $= \log_2 e$ bits.

Wiener, Shannon, and Fano[14] give a number of reasons for the use of this function as a measure of information per symbol, and the arguments are plausible and satisfying, but as Shannon remarks, the ultimate justification of the definition is in the implications and applications of entropy as a measure of information.[15] For the analysis of communications systems, the definition is completely justified by theorems which prove that it is possible to code any message with entropy H bits per symbol in a binary code which uses an average of $H + \epsilon$ binary digits per message symbol, where ϵ is a positive quantity which may be made as small as desired, and by equivalent theorems in the case of the discrete channel with noise —i.e., where there is a finite probability that a symbol

[14] This is the definition given by Shannon (Shannon and Weaver, *op. cit.*) and Fano (R. M. Fano, "The Transmission of Information", Tech. Rep. No. 65, Res. Lab. Elec., M. I. T.; 1949) Wienes ("Cybernetics", *op. cit.*, p. 76) gives a definition with the opposite sign. There is no real conflict here, however, for Wiener is talking about a different measure. Wiener asks, how much information we are given about a message term, whose exact value will never be known, when we are given the probability distribution from which it is chosen. The answer is that we know a good deal when the distribution is narrow, and very little when the distribution is broad. Correspondingly, entropy as Wiener defines it has a large positive value for very narrow distributions and a large negative value for very broad distributions. This measure is useful in determining how much information has been transmitted when a message term which is contaminated by noise with a known distribution is received; we can use Bayes' theorem and find the probability distribution of the original message, and measure information transmitted by measuring the entropy of this distribution. Shannon, on the other hand, asks how much information is transmitted by the precise transmission of a message symbol, when we know *a priori* the probability distribution from which it was selected. In this case, if the distribution is very narrow, the message term tells us very little when it arrives; we knew what it would be before we received it. If the distribution is broad, however, then the arrival of the term tells us a good deal. This requires the use of the opposite sign for entropy. Shannon's definition will be used through this paper; it is the more appropriate one for the kind of problem with which we are concerned.

[15] Shannon and Weaver, *op. cit.*, p. 19.

transmitted at one quantized level will be received at a different level, and in the case of the continuous channel with noise—in which the message term is chosen from a continuous distribution, and is received mixed with noise, so that each received term is the sum of a signal term and a noise term, and reception is always approximate.

For messages as defined in a previous section, we have, in general, that the entropy of the distribution from which any single message term is chosen is a function of the message history: in both the continuous and discrete cases we are concerned with conditional distributions, whose form depends on the set of values of the terms $m_{i-1} \cdots m_{i-j} \cdots$ which prcede the message term m_i whose entropy is defined in (8). For such cases—i.e., Markoff processes of order one or greater—the entropy is defined in terms of the probability, not of each message term, but of a sequence of N message terms, the limit being taken as N approaches infinity. Following Shannon,[16] we define G_N in the discrete and the continuous cases as

$$G_N = -(1/N) \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} M(m_i, \cdots m_{i-N}) \cdot \log M(m_i, \cdots m_{i-N})\, dm_i \cdots dm_{i-N}$$

$$= -(1/N) \sum_{m_i=-\infty}^{\infty} \cdots \sum_{m_{i-N}=-\infty}^{\infty} M(m_i, \cdots m_{i-N}) \cdot \log M(m_i, \cdots m_{i-N}). \quad (9)$$

Then the entropy per symbol of the process is defined as

$$H = \lim_{N\to\infty} G_N. \quad (10)$$

The distribution $M(m_i, \cdots m_{i-N})$ in (9) is not a conditional but a joint distribution: the distribution which determines the probability of getting a given set of N values for the $N + 1$ message terms m_{i-N} to m_i. Now the joint probability distribution of order $N + 1$ is related to the conditional probability distribution and the joint distribution of order N by

$$M(m_i, \cdots m_{i-N}) = M(m_i : m_{i-1} \cdots m_{i-N}) M(m_{i-1}, \cdots m_{i-N}). \quad (11)$$

Using the relation (11) in the expression (9), for a message generating process which is a Markoff process of finite order k, and taking the limit (10), we have

$$H = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} M(m_{i-1}, \cdots m_{i-k})\, dm_{i-1} \cdots dm_{i-k} \cdot \left\{ \int_{-\infty}^{\infty} M(m_i : m_{i-1} \cdots m_{i-k}) \cdot \log M(m_i : m_{i-1} \cdots m_{i-k})\, dm_i \right\} \quad (12)$$

with a similar relation for the discrete case, in which the integrals are replaced by sums. In words, what (12) states is that the entropy for the process as a whole is just the average over-all past histories of the entropy of the conditional distribution of order k which defines the process: the information content per symbol of a message generated by such a stochastic process is the average of the entropies of the distribution from which the successive message terms are chosen.

It was noted that only a finite order Markoff process can, in general, be used as a model of a message source, and that, in general, the use of such an approximation is inefficient. We may now state this more exactly. If a kth order Markoff process is approximated by a process of order less than k, then the entropy of the approximating process will be greater than or equal to the entropy of the original process, with the equality holding only if the original process is actually of order less than k: i.e., only if the kth order conditional distribution can be expressed in terms of conditional distributions of lower order. The result holds also for suitably convergent processes of infinite order. It is a direct consequence of the following more general theorem.

Averaging Theorem I

Let $P(x: y)$ be a probability density distribution of x, for each value of the parameter y: i.e., for all y,

$$\int_{-\infty}^{\infty} P(x: y)\, dx = 1,$$

and

$$P(x: y) \geq 0$$

for all x and y. Let $Q(y)$ be a probability density distribution of y:

$$\int_{-\infty}^{\infty} Q(y)\, dy = 1$$

$$Q(y) \geq 0.$$

Let $R(x)$ be the distribution $P(x: y)$ averaged over the parameter y, and let H' be its entropy:

$$R(x) = \int_{-\infty}^{\infty} Q(y) P(x: y)\, dy$$

$$H' = -\int_{-\infty}^{\infty} R(x) \log R(x)\, dx. \quad (13)$$

Let $H(y)$ be the entropy of the distribution $P(x: y)$ as a function of the parameter y, and let H be its average value:

$$H(y) = -\int_{-\infty}^{\infty} P(x: y) \log P(x: y)\, dx$$

$$H = \int_{-\infty}^{\infty} Q(y) H(y)\, dy.$$

Then we always have $H' \geq H$, and the equality holds only when the y dependence of $P(x: y)$ is fictitious. In words, the entropy of the average distribution is always greater than the average of the entropy of the distribution.

[16] Shannon and Weaver, *op. cit.*, p. 25.

The proof is given in the appendix.[17] The theorem remains true for discrete distributions, and the statement is unchanged except for the uniform substitution of the summation indexes i and j for the continuous variables x and y and the replacement of integrations by sums. By successive application of the proof it is also obvious that the result holds for a distribution which is a function of n parameters y_1 to y_n. The application to Markoff processes is direct, for a conditional distribution of order $k - 1$ may be expressed as an integral of the form $R(x)$ in (13), where $P(x: y)$ is the conditional distribution of order k and y is the term m_{i-k}.

The theorem is also applicable to cases in which the dependence of the distribution on past history is not explicit. If the dependence of the distribution $M(m_i : m_{i-1} \cdots m_{i-k})$ on the set of past message values is through a dependence on one or several parameters (e.g., the mean and the standard deviation of a distribution are functions of the set of past message values but the distribution is always normal), the conclusion still holds: the entropy of the average distribution, averaged over the distribution of the parameters, is always greater than the average over the parameters of the entropy. This is illustrated by the example of (4). The average message term distribution of the process is a normal distribution with a standard deviation $\sigma/\sqrt{1 - a^2}$, with an entropy which may easily be computed[18] as

$$H_0 = \log \sigma\sqrt{2\pi e} + \log (1/\sqrt{1 - a^2}),$$

but each message term has a normal distribution with standard deviation, with entropy just

$$H = \log \sigma\sqrt{2\pi e},$$

which is thus the average entropy of the process as a whole. The difference between these two entropies may be made as large as we like by letting a approach one.

A second averaging theorem which will be useful later deals with averages over a single distribution.

Averaging Theorem II

Let $P(x)$ be a probability distribution with entropy H:

$$\int_{-\infty}^{\infty} P(x)\, dx = 1, \qquad P(x) \geq 0 \qquad \text{for all } x,$$

$$H = -\int_{-\infty}^{\infty} P(x) \log P(x)\, dx.$$

Let $Q(x, y)$ be a weighting function:

$$\int_{-\infty}^{\infty} Q(x, y)\, dx = \int_{-\infty}^{\infty} Q(x, y)\, dy = 1,$$

$$Q(x, y) \geq 0 \qquad \text{for all } x \text{ and } y.$$

Let $R(x)$ be the averaged distribution with entropy H':

$$R(x) = \int_{-\infty}^{\infty} P(y)Q(x, y)\, dy$$

$$H' = -\int_{-\infty}^{\infty} R(x) \log R(x)\, dx.$$

Then we always have $H' \geq H$, and the equality holds only when the weighting function is a Dirac delta function.

This theorem is given by Shannon.[19] It is also true in the discrete case: the equality then holds only if the average distribution $R(x)$, or R_i in the discrete case, is a mere permutation of the distribution $P(x)$, or P_i.

At the beginning of this section it was stated that it is possible to code a message with entropy H bits per symbol by a coding method which uses $H + \epsilon$ binary output symbols per input symbol, on an average. Such a coding scheme will be called an *ideal code*. Shannon has given two such coding procedures, and Fano has given one which is quite similar to one of Shannon's.[20] We will call coding by means of Shannon's second procedure, or by means of Fano's method, *Shannon-Fano coding*. Both are procedures for giving short codes to common messages and long codes to rare messages. They are given in the references. We will here only note the important result. Coding a group of N message terms at once, the average number H_1 of output binary symbols per input message symbol is bounded:

$$G_N \leq H_1 \leq G_N + 1/N. \tag{14}$$

Here G_N is the quantity defined in (9). As N increases G_N approaches H, the true entropy of the process, so H_1 also approaches H. For a discrete process, an *efficient* code may be defined as one for which the ratio H/H_1 is near one. It is clear that there are two reasons why a Shannon-Fano code for small N may be inefficient: first, if G_N is small, the ratio G_N/H_1 may be small, if H_1 is near its upper bound in (14). Second, for small N, G_N may be a poor approximation to H.

It should be noted that it is *not* reasonable to define an efficiency measure for continuous distributions as a ratio of entropies. For a process which is ultimately to be quantized, the entropy of a continuous distribution does not approximate the entropy of the discrete distribution which is obtained by quantization, unless the scale of the variable in the continuous distribution is so chosen as to make the interval between quantized levels unity. Using a different choice of scale adds a constant to the entropy of the distribution, so that the ratio which defines efficiency is changed. For this reason, until a quantizing level spacing is chosen, it is possible to speak only of the differences between the entropies of continuous distributions, and not of their ratios.

[17] The content of this theorem is implied by the derivations leading up to Shannon's fundamental theorem, Shannon and Weaver, *op. cit.*, p. 28. However, the theorem can be stated and proved as a property of entropy as a functional of a probability distribution, with no reference to sequences of message terms, and the proof is so straightforward and simple that the theorem deserves an independent statement.

[18] Shannon and Weaver, *op. cit.*, p. 56.

[19] Shannon and Weaver, *op. cit.*, p. 21, property 4 for the discrete case; p. 55, property 3 for the continuous case.

[20] Shannon and Weaver, *op. cit.*, p. 29; Fano, *op. cit.* Shannon's procedure is simpler to handle mathematically; Fano's is perhaps somewhat simpler to grasp. Fano's method is not quite completely determinate. In cases in which the two methods do not agree, Fano's provides a more efficient code than Shannon's.

Appendix

Proof of Averaging Theorem I

Expanding H' by the definitions given, we have

$$H' = -\int_{-\infty}^{\infty} dx \left\{\int_{-\infty}^{\infty} dy\, Q(y)P(x:y) \log \int_{-\infty}^{\infty} dz\, Q(z)P(x:z)\right\}.$$

Adding and subtracting a term gives

$$H' = -\int_{-\infty}^{\infty} dx \left\{\int_{-\infty}^{\infty} dy\, Q(y)P(x:y) \log P(x:y) + \int_{-\infty}^{\infty} dy\, Q(y)P(x:y) \log \frac{\int_{-\infty}^{\infty} dz\, Q(z)P(x:z)}{P(x:y)}\right\}.$$

The quotient in the last integral cannot cause trouble, since the integrand as a whole approaches zero with $P(x:y)$. Interchanging the order of integration in the first integral and using the definition of H gives

$$H' = H - \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, Q(y)P(x:y) \cdot \log \frac{\int_{-\infty}^{\infty} dz\, Q(z)P(x:z)}{P(x:y)}. \qquad (14)$$

Changing the logarithmic base will multiply both sides of (14) by the same constant, so we are free to use natural logarithms and measure entropy in natural units. Using the inequality $\log u \leq u - 1$ in the integral in (14) gives

$$H' \geq H - \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, Q(y)P(x:y) \cdot \left\{-1 + \frac{\int_{-\infty}^{\infty} dz\, Q(z)P(x:z)}{P(x:y)}\right\}$$

$$\geq H + \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, Q(y)P(x:y) - \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz\, Q(y)Q(z)P(x:z).$$

Integrating first with respect to x, we have by the normalization requirements on $P(x:y)$ and $Q(y)$ that

$$H' \geq H + 1 - 1 \geq H.$$

The equality can be realized only when $\log u \equiv 1$, or in this case when

$$\int_{-\infty}^{\infty} P(x:z)Q(z)\, dz \equiv P(x:y). \qquad (15)$$

For this to hold, $P(x:y)$ must have no dependence on variable y, since y does not appear on the left of (15), Q.E.D. In the discrete case, the precise same proof holds when summations are uniformly substituted for integrations.

Predictive Coding—Part II

Summary—**In Part I predictive coding was defined and messages, prediction, entropy, and ideal coding were discussed. In the present paper the criterion to be used for predictors for the purpose of predictive coding is defined: that predictor is optimum in the information theory (IT) sense which minimizes the entropy of the average error-term distribution. Ordered averages of distributions are defined and it is shown that if a predictor gives an ordered average error term distribution it will be a best IT predictor. Special classes of messages are considered for which a best IT predictor can easily be found, and some examples are given.**

The error terms which are transmitted in predictive coding are treated as if they were statistically independent. If this is indeed the case, or a good approximation, then it is still necessary to show that sequences of message terms which are statistically independent may always be coded efficiently, without impractically large memory requirements, in order to show that predictive coding may be practical and efficient in such cases. This is done in the final section of this paper.

Definition of Information-Theory Criterion for Predictors

We have now a sufficient vocabulary and collection of results to define and discuss a criterion of prediction that is appropriate for the kind of communications scheme outlined in the Introduction. An obvious definition is: that predictor is best, in the sense of information theory, which requires the minimum channel space for the transmission of its error term. But this specification is not yet sufficient. It is necessary to define to some extent the way in which the error term is to be coded, in order to define a predictor uniquely for a given message-generating process.

One procedure is to use Shannon-Fano coding for the transmission of the error term. This means that the predictor $p(m_{i-1} \cdots m_{i-j} \cdots)$ should be chosen to minimize the ensemble average of the entropy of the error distribution. The average of

$$-\int_{-\infty}^{\infty} E(m_i : m_{i-1} \cdots m_{i-J} \cdots) \cdot \log E(m_i : m_{i-1} \cdots m_{i-j} \cdots)\, dm_i \qquad (16)$$

or of

$$-\sum_{-\infty}^{\infty} E_k(m_{i-1} \cdots m_{i-j} \cdots) \log E_k(m_{i-1} \cdots m_{i-j} \cdots)$$

is to be minimized, the averaging being done over the

message term joint distribution $M(m_{i-1} \cdots m_{i-j} \cdots)$.[21] This procedure, however, does not lead to a determination of the predictor p at all. Indeed, the expressions (16) are quite independent of p. For as pointed out in Part I, the distribution of the error term e_i is the same as the distribution of the message term m_i except for a shift in the mean value, i.e., a translation of the function along the axis of the independent variable. Now the entropy of a distribution is quite independent of any such shift. It is a sort of measure of the dispersion of the distribution, and not of the location of its mean. So the number obtained by averaging (16) over the message term distribution will, for each predictor, have the same value—a value which is precisely the information content of the original message. It is clear that this must be so if entropy is to be a valid measure of information. The information contained in the error terms is precisely all of the information present in the original message, since either can be obtained unambigously from the other; and the transmission of either, by an ideal code, will take the same amount of channel space.

There is, however, a significant information theory criterion for predictors. An efficient Shannon-Fano code for a highly predictable message-generating process requires, as discussed in Part I, that a large number N of terms be remembered and coded at one time, so that the average number of bits per symbol in the coded message H_N', which is nearly equal to G_N, will be near the true entropy of the process H. This imposes two requirements on the coding process.

1. There must be an active memory which remembers the past N message terms.

2. There must be a translator, or codebook memory, which remembers the codes for all possible groups of N message terms. If each term is chosen from an M-ary discrete distribution, there must be N^M entries in the codebook.

For the transmission of a written message, these requirements are not too difficult to meet. The message itself, available at the transmitter, meets the first, and a codebook of the sort used for the commerical cable codes meets the second to a degree, with not too much loss of time in the coding process. But for an automatic device transmitting a rapidly occurring message, with sampling intervals of the order of microseconds or less, requirement 2 becomes highly impractical for many messages in which M and N may both be larger than ten.

The usual solution to this problem is to ignore the correlations between successive message terms, and to code the message as if it were a zero-order Markoff process, in which each message term m_i is chosen from the average message term distribution $\overline{M}(m_i)$. In this treatment the channel space required for the transmission of the message is increased, as is shown in Part I, and is determined by the entropy of the average message term distribution, rather than by the true entropy of the process, which is the average entropy of the message term distribution. There is then no active memory requirement, and the codebook requirement of M codes for the M possible message levels is manageable in size, and may be realized by such devices as the binary coding tube.[22]

Predictive coding may be used as an intermediate step between the two limits of ideal and zero-order coding. As in zero-order coding, the correlation between successive terms of the transmitted message is ignored; but the transmitted message is now the error term and not the original message. The channel capacity required by this procedure is then equal to the entropy of the average error term distribution, if we can find an efficient method of coding the error terms which does not require a large codebook. Such a coding method will be called *nonmnemonic*. It will be shown that this can be done.

These considerations suggest the following definition. Let $\bar{E}$ be the average error distribution corresponding to a particular predictor p:

$$\bar{E}(e_i, p) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} E(e_i : m_{i-1} \cdots m_{i-j} \cdots) \cdot M(m_{i-1} \cdots m_{i-j} \cdots)\, dm_{i-1} \cdots dm_{i-j}$$

$$\bar{E}_k(p) = \sum_{m_{i-j}=-\infty}^{\infty} \cdots \sum_{m_{i-j}=-\infty}^{\infty} E_k(m_{i-1} \cdots m_{i-j} \cdots) \cdot M(m_{i-1} \cdots m_{i-j} \cdots), \qquad (17)$$

where M is the joint distribution of the prior message values as before. Then:

Definition

The best predictor, in the information theory sense, is that function $p(m_{i-1} \cdots m_{i-n} \cdots)$ which minimizes the entropy of the averaged error distribution, $\bar{E}(e_i, p)$ or $\bar{E}_k(p)$.

The entropy of $\bar{E}$ is by no means independent of the predictor p; a number of different distributions are being averaged, and in simple cases it is obvious that, if the distributions are narrow, the resultant curve will be broad if the means of the averaged distributions are widely dispersed, and narrow if they are near to one another. Entropy is a functional of a distribution which is large when the distribution is broad and small when it is narrow, and it is now reasonable to ask how to choose the means of the otherwise determined distributions which are being averaged, so as to minimize the entropy of the resultant average distribution.

There are several general points which may be noted about this criterion of prediction.

1. The addition of a constant to a best IT (Information Theory) predictor will give a best IT predictor. The

[21] Although the discussion will continue to include both the discrete and the continuous cases, the reader is reminded that the continuous distributions are used for mathematical convenience only, and ultimate quantization is always implied. Therefore the transmitted message is always discrete, and the Shannon-Fano coding method cited in Part I may always be employed.

[22] R. W. Sears, "Electron beam tube for pulse code modulation," *Bell Sys. Tech. Jour.*, vol. 27, pp. 44–57; January, 1948.

predictor is determined by this criterion only up to an additive constant. For, if the means of all of the distributions being averaged are shifted by the same amount, the resultant will have its mean shifted by an equal amount, but will not be otherwise changed, and the entropy is invariant under such shifts.

2. Even if the additive constant has been determined by some additional constraint (such as the requirement that the mean of $\bar{E}$ be zero), the predictor may not be unique. An example will appear later. Here we only note that, while this will be rare in practical cases, there will be many cases in which there is a large class of predictors with very nearly the same entropy.

3. The transmission method presented in Part I, using a predictor which satisfies this criterion, will not in general be ideal. It will be ideal in some important special cases, and it will be nearly ideal, and much more efficient than zero-order coding, in cases in which the error distribution is narrow for any past history but has a mean which changes widely as the recent history changes. In such cases the IT criterion will lead to predictors which are close to the best rms or other moment criterion predictors.

From a practical point of view, this method is much more feasible in highly predictable cases than is the Shannon-Fano coding. As will be shown in an example, the active memory requirement may be much smaller; far more important, the codebook required is always small. Coding methods will be considered later in detail. It turns out that the codebook memory will rarely have more than M entries, and that when it is necessary to code error terms in groups, non-mnemonic coding may always be used.

Unfortunately no elegant and general solution for the best IT predictor has been found, and there is reason to believe that it will not be possible to find such a solution in which no restrictions are placed on the message other than those listed in Part I. However, it is possible to find the best IT predictor when suitable additional restrictions are imposed. We will consider several sets of restrictions, which seem likely to cover most of the cases in which a quantized message is obtained by sampling a smoothly-varying function of time. In some of these the best IT predictor is obtained explicitly, but for certain significant cases approximation methods must be used.

Ordered Averages and Special Cases

Predictive coding involves the averaging together of a number of error term distribution. As shown in Part I, such an averaging in general increases the entropy. An *ordered average* is a kind of average which minimizes this increase. It is not always possible to obtain an ordered average by using predictive coding, but when it is possible to do so, the predictor used will be a best IT predictor. In any case, the entropy of the ordered average may be computed from the statistics of the message, if these are sufficiently well known, and it provides a lower bound to the savings in channel space which may be achieved by the use of predictive coding. It is thus useful in deciding whether or not it is worthwhile to investigage predictive coding for application to a particular kind of message.

Consider a set of n discrete distributions, each having a total of m terms, and let M_{jk} represent the probability, in the jth of these distributions, of the occurrence of the kth symbol. For a particular set of averaging weights a_j with

$$a_j \geq 0$$

$$\sum_{j=1}^{n} a_j = 1,$$

the ordered average is defined by taking a_1 times the largest probability in the first distribution, a_2 times the largest probability in the second distribution, and so on. The weighted average of these largest probabilities is the first term, $\vec{M}_1$, in the ordered average. Using the same weights and the second largest probabilities in each distribution gives the ordered average term $\vec{M}_2$, and so on to the average of the smallest probabilities, which gives $\vec{M}_m$.

It can be shown that the entropy of the ordered average distribution is less than that of any other average formed from the same set of probability distributions with the same weights, but with the terms of one or more of the distributions not arranged in order of decreasing probability.[23] There are, therefore, in general, four different entropies to be considered in the predictive coding of a message: H_0, the entropy of the average message term distribution; H', the entropy of the average error term distribution using the best IT predictor; H'', the entropy of the ordered average error term distribution; and H, the true entropy of the message-generating process. Clearly,

$$H \leq H'' \leq H' \leq H_0 . \tag{18}$$

The ordered average thus gives a lower bound to the entropy H', which may be greater than the true entropy of the process H. The equality signs in (18) are all necessary; in a message-generating process in which each term is selected independently from a constant distribution, all four entropies are the same. We will next discuss three cases in which $H' = H''$. In one of these $H' = H$ as well, and predictive coding may therefore be ideal.

Case I

Let the message term m_i be drawn from a distribution which has always the same form, but with a mean which is determined by the past message values: i.e.,

$$M(m_i : m_{i-1} \cdots m_{i-j} \cdots) = M(m_i - p[m_{i-1} \cdots m_{i-j} \cdots]). \tag{19}$$

In this case, it is clear that averaging with coincident means leads to an ordered average, indeed an average error term distribution of the same form as the original message term distribution. We have the following properties:

1. The best IT predictor is defined uniquely (except for an additive constant) and is $p(m_{i-1} \cdots m_{i-j} \cdots)$. It yields an ordered average: in the expression (18), $H'' = H'$.

2. The best IT predictor is the same as the best rms predictor, and indeed is the same as the best absolute nth moment predictor.

[23] P. Elias, "Predictive Coding," Thesis, Harvard Univ. Press, Cambridge, Mass.; May 1, 1950.

3. Predictive coding using the best IT predictor can be ideal: in the expression (18), $H' = H$.

This case sounds quite special. Actually, however, it includes all of the message-generating processes for which Wiener's filters and predictors were specifically designed—i.e., the responses of linear resonators to Brownian motions—and many more processes as well, in which the predictor is not linear, or in which the distribution of the error term is not Gaussian, or both. It also approximates a large number of processes of physical interest, in which the message varies slowly (relative to the sampling time interval) and more or less continuously, and the successive error terms are chosen from distributions which are not identical, but are similar. As the distributions approach identity, properties of the process approach the three listed above, hence these are good approximations in such cases.

Case II

Let the message term m_i be drawn from a distribution which is symmetric and unimodal,[24] but is otherwise an arbitrary function of the past history of the message:

$$M(p + x: m_{i-1} \cdots m_{i-j} \cdots) = M(p - x: m_{i-1} \cdots m_{i-j} \cdots)$$

$$M(p + x: m_{i-1} \cdots m_{i-j} \cdots) > M(p + y: m_{i-1} \cdots m_{i-j} \cdots) \qquad (20)$$

for $|x| < |y|$, with

$$p = p(m_{i-1} \cdots m_{i-j} \cdots).$$

Here it is again evident that averaging with coincident means will give an ordered average. We still have properties (1) and (2) listed under Case I: it is clear that the IT and absolute moment predictors agree. However, property (3) has been lost. The distributions being averaged are not in general identical, and the entropy of the averaged error term distribution, by the averaging theorem of Part I, is in general greater than the average of the entropies of the message term distributions, which is the entropy per symbol of the original message-generating process.

Physically, this is a very broad category of processes. It may be expected to include most messages derived by the time-sampling of a smoothly varying function generated by a process which is essentially symmetrical. The Brownian processes are again included.

Case III

Let the message term be drawn from a distribution which is zero to the left (right) of a point p, is discontinuous at that point, and is monotonic nonincreasing to the right (left):

$$M(m_i : m_{i-1} \cdots m_{i-j} \cdots) = 0,$$

for

$$m_i < p(m_{i-1} \cdots m_{i-j} \cdots)$$

$$M(x: m_{i-1} \cdots m_{i-j} \cdots) \geq M(y: m_{i-1} \cdots m_{i-j} \cdots)$$

for

$$x < y. \qquad (21)$$

The point p may be an arbitrary function of the message history. Clearly, averaging the error term distributions with the points p coincident will give an ordered average. Here we have only the first of the three properties listed under Case I: the IT predictor does not agree with the absolute moment predictors, and in general ideal coding cannot be obtained.

This category is physically important, since there are many physical processes which are of an essentially positive nature. However, it is not to be expected that predictive coding will be very useful for processes of this sort. For the point p is not usually a function of the message history, but is a fixed origin of some kind, and the message as originally generated will have the points p already coincident—the best IT predictor will be zero or any constant, and the entropy of the average error term will be the entropy of the average message distribution: in the expression (18), $H_0 = H' = H''$.

These three cases exhaust the processes of physical interest for which a predictor yielding an ordered average can be obtained. There are additional cases in which this can be done, but they are quite special and not likely to correspond to a message which has been derived by time sampling of a smoothly varying function of time. Indeed, most processes of physical interest as possible messages derived from such sampling are included in the above three cases or in one other, for which the best IT predictor can usually be found only by approximation methods.

Case IV

Let the message term m_i be chosen from a distribution which has the symmetry property that:

$$M(m_i : m_{i-1} \cdots m_{i-j} \cdots) = M(-m_i : -m_{i-1} \cdots -m_{i-j} \cdots). \qquad (22)$$

That is, reflecting the past history terms across the time axis causes the reflection of the distribution about the time axis. Here we have none of the properties listed in Case I: the four quantities in (3) are all distinct, and the best IT predictor cannot be explicitly found.

Mathematically, the class of processes in this category is also quite broad; the Brownian processes are again included. The physically interesting new cases included are those in which some kind of limiting action is present at the source of the message, so that the distribution from which a message term is selected is more or less skewed towards the time axis. Such a situation arises, for example, when a normally distributed message term is passed through a symmetric nonlinear device. The transfer characteristic of such a device, and its effect on a normal message term distribution, is illustrated in Fig. 2 (next page). Actually, any electronic system will have some such nonlinear limiting—although not necessarily symmetrical—for sufficiently large signal amplitudes, but the linear range may be so large that the limiting is not evident.

It should be noted that an average message distribution

[24] Unimodality is defined for our purposes by (20). It means, not merely only one peak that reaches the maximum value, but only

amples of Part I yield an average message distribution with finite power. Another example is a process in which the message term m_i is chosen from a normal distribution with fixed standard deviation and with a mean which is $m_{i-1}/(1 + |m_{i-1}|)$. This process is symmetrically limited and is included under Case IV, but it is also included under Cases I and II, so that the best IT predictor is immediately determinable.

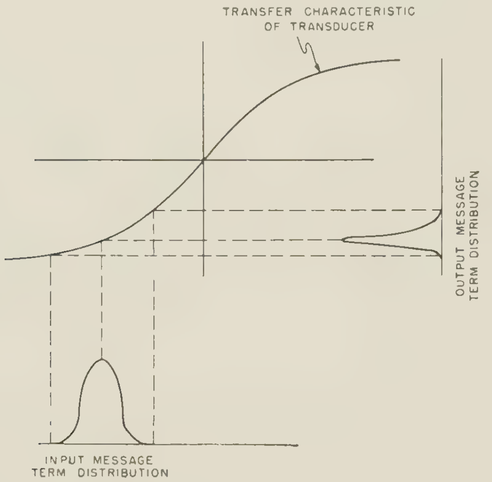

Fig. 2—The effect of limiting on a probability distribution.

Although in Case IV the best IT predictor, in general, cannot be found explicitly, for practical purposes the situation is not bad. For, if the distribution of each message term is broad, and the differences for different past histories are largely differences in skewing, and not in the location of the mean, then the savings to be expected using predictive coding, even with the best IT predictor, will not be large. If, on the other hand, the difference in distributions is largely in the location of the mean, and the distributions are narrow, then predictive coding will be useful in reducing the channel space required as compared to zero-order coding, but almost any reasonable predictor will be almost as good as the best IT predictor. This will be shown in the next section by an example.

Examples of Predictable Processes

Example I

As a first example, consider the Brownian process given in Part I, with the continuous distribution

$$M(m_i : m_{i-1}) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-(m_i - am_{i-1})^2/2\sigma^2\right]$$

and

$$\overline{M}(m_i) = \frac{1}{\sigma'\sqrt{2\pi}} \exp\left[-m_i^2/2\sigma'^2\right], \text{ with } \sigma' = \sigma/\sqrt{1 - a^2}.$$

This process is an example for Cases I, II, and IV. It therefore has the three properties listed for Case I. As discussed in Part I, the entropy H of the process is less than the entropy H_0 of the zero-order approximation to it by an amount $\log(1/\sqrt{1 - a^2})$: by property (3) of Case I, this full savings may be realized by using predictive coding. For a near one the savings may be appreciable.

Example II

Consider the discrete conditional distribution

$$\begin{aligned} M_{kj} &= 0 \qquad \text{for} \quad k < 0 \quad \text{or} \quad k > j + 1 \\ &= \frac{1}{j + 2} \qquad \text{for} \quad 0 \leq k \leq j + 1, \end{aligned}$$

where j is the value of the preceding message term m_{i-1}, and M_{kj} is the probability that m_i will take the integer value k after m_{i-1} has taken the value j. Each message term is here chosen from a distribution which gives equal probability to all the integers from 0 to the integer one greater than the preceding message term.

From the consistency requirement on the average distribution $\overline{M}_k$,

$$\overline{M}_k = \sum_{j=0}^{\infty} M_{kj}\overline{M}_j \tag{23}$$

it is possible to find $\overline{M}_k$: the result is the Poisson distribution with $\lambda = 1$,

$$\overline{M}_k = (1/e)1/k! \tag{24}$$

This process is an example of Case III. As indicated above, since the origin of the process is independent of the previous message terms, a best IT predictor is zero or any constant; predictive coding and zero-order coding will amount to the same thing, and the channel space required will be that required by a source whose entropy per symbol is the entropy of $\overline{M}_k$. The process also indicates the possible ambiguity of the IT criterion for predictors. Since the distributions being averaged are all flat, there is a rather wide range of predictors which will yield an ordered average error term. In particular, the means of the distributions (or one of the two integers nearest to the mean in the case of even j) may be aligned, giving the best rms predictor, and the resultant average distribution will have the same monotonic normal form as does $\overline{M}_k$, and thus the same entropy. The best rms predictor is thus a best IT predictor. This will not be true for typical examples of Case III, in which the distributions being averaged are strictly monotonic: in such processes there is no ambiguity.

Example III

Consider the discrete conditional distribution described by

$$\begin{aligned} M_{kj} &= b(1 + j/N), \quad k = j - 1 \\ &1 - 2b, \qquad k = j \\ &b(1 - j/N), \quad k = j + 1 \\ &0, \qquad k < j - 1 \quad \text{or} \quad k > j + 1. \end{aligned} \tag{25}$$

This is a symmetrically bounded process, in which m_i cannot exceed N. The average distribution $\overline{M}_k$ is again

determined by the consistency requirement (23): it turns out to be independent of the parameter b:

$$\overline{M}_k = (1/2^{2N})(2N)!/(N+k)!(N-k)! \qquad (26)$$

which is the bionomial distribution of order $2N$.

Using m_{i-1} as a predictor for m_i, the average error term distribution $\bar{E}_k$ turns out to be a three-term distribution:

$$\begin{aligned} \bar{E}_k = b, &\qquad k = -1 \\ 1-2b, &\qquad k = 0 \\ b, &\qquad k = +1 \\ 0, &\qquad k < -1 \quad \text{or} \quad k > +1. \end{aligned} \qquad (27)$$

The nine conditional distributions for $N = 4$, and the average error distribution, are shown in Fig. 3. Here the parameter b has been taken as 1/4. For this value of b, the entropy of the average error distribution $\bar{E}_k$ is 1.5 bits.

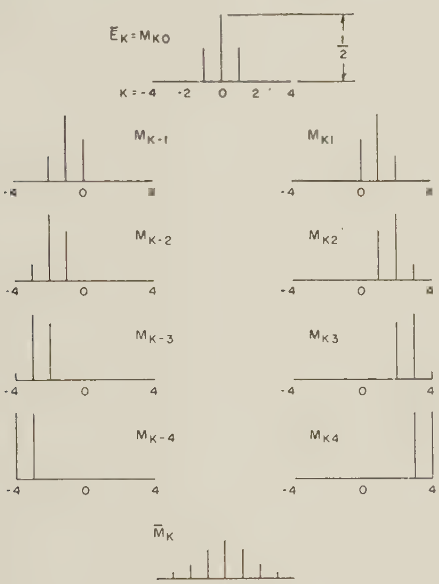

Fig. 3—Message term and error distributions for Example III.

The entropy of the binomial distribution of order $2N$ is very nearly $\frac{1}{2}\log_2 N + 1\frac{1}{2}$ bits, and the savings in channel space which may be obtained by using predictive coding rather than zero-order coding, is given by R, the ratio of these two entropies:

$$R \doteq \frac{3}{3 + \log_2 N}$$

For smaller values of b, the ratio is still smaller. For H_E, the entropy of the average error distribution, we have the approximate expression

$$\begin{aligned} H_E &= -2b \log_2 b - (1-2b) \log_2 (1-2b) \\ &\doteq -2b \log_2 b + 2b(1-2b) \log_2 e, \qquad b \ll 1 \\ &\doteq 2b \log_2 \frac{e}{b}. \end{aligned}$$

This gives

$$R \doteq \frac{4b \log_2 (e/b)}{3 + \log_2 N}, \qquad b \ll 1.$$

This process is an example of Case IV. The predictor $m_i = m_{i-1}$ does not yield an ordered average. However, the difference between the entropy of the average error term, using this predictor, and the entropy of an ordered average error term, is small for large N. Indeed, for $b = \frac{1}{4}$ and $N = 1$, the entropy of $\bar{E}_k$ is 1.5 and the entropy of an ordered average is 1.406, so that the difference is already of little practical importance. It may also be shown that predictive coding is asymptotically ideal for this process for large N; for large N the process becomes very much like a process in Case I, in which all terms are drawn from identical error distributions, for the probability of m_{i-1} being very far from 0 becomes very small, and if m_{i-1} is near 0 then the distribution of m_i is nearly the symmetric three-term distribution of (27).

The properties of this process for large N show the meaning of the statement made in the discussion of Case I, to the effect that distributions which approximated the condition (19) also approximated the properties (1), (2), and (3). It also illustrates the savings in active memory. If b is quite small, there will be long sequences of successive message terms which are the same value. For Shannon-Fano coding to be efficient, it will be necessary to take a sequence of message terms so long as to be representative of the process, and the active memory requirement will thus be very large. However, with predictive coding, only one message term need be remembered to give a good IT prediction as to the value of the next term. This is perhaps clearer in connection with this example, which is already quantized, than it is in connection with the continuous process of Example I.

The examples given have all been Markoff processes of the first order. Interesting higher-order processes are easy to define, but it is very difficult to determine the average message distribution for them, and to evaluate the entropy of the process, the entropy of the average message distribution, and the entropy of the average error term distribution for a reasonable predictor. This sort of computation is possible for a Brownian process and for virtually no others, except by numerical methods. However, this is not too serious a handicap for the application of predictive coding, for there are very few message-generating processes which are known *a priori*. The significant practical problem is to measure the statistics of the process and find the best IT predictor or a reasonable approximation to it. This will be difficult if more than one or two past message values significantly affect the predictor, as discussed in Part I. However, an experimental variational approach may be used, in which a trial predictor is selected and the average error term which results from the use of this predictor is measured. The predictor is varied to minimize the entropy of the resultant average error distribution. Not only is this a method for finding the best IT predictor for predictive coding; it is also a

method for finding an upper bound to the entropy of the process being measured. Using the relations (18) from right to left, we have $H' \geq H'' \geq H$. The entropy of the average error distribution of a trial predictor, H_E, is larger than the entropy of the best IT predictor, H', by definition, so for any trial predictor, H_E is an upper bound for the entropy H of the message-generating process. While the upper bound provided may not be very good, for such processes as television and facsimile it would be very helpful, since there are at present available only widely divergent guesses as to the entropy per symbol of these processes.

Coding Methods for Independent Message Terms

Predictive coding treats successive error terms as if they were uncorrelated: i.e., as if the only correlation between successive message terms were due to changes in the mean, and not in the form, of the message term distribution. As we have seen in the preceding examples, this procedure is sometimes ideal and often efficient. That is, it leads to an average error term distribution which, in many cases of interest, may be expected to have only slightly more entropy than the true entropy per symbol of the original message-generating process. This leads us to a consideration of the general problem of the efficient coding of a message having uncorrelated terms. The discussion will be restricted to quantized messages. For practical purposes, we are interested in coding methods which either are non-nmemonic or required only small codebook memories. It turns out that the problem can be solved in one or another of two ways, depending on the distribution from which the uncorrelated message terms are chosen.

Case I: $H > 1$

If H, the entropy of the message term distribution, is large compared to one bit, then Shannon-Fano coding of each message term will be efficient. For from (14) in Part I, we have

$$H \leq H_1 \leq H + 1, \tag{28}$$

since $G_N = H$ for a message with uncorrelated terms. This gives an efficiency $R \geq H/(1 + H)$ which is near one for $H \gg 1$. Since we are coding only one message term at a time, the codebook memory has only M entries for the M possible quantized values of the message, and may conveniently be realized by such devices as the binary coding tube[22]. Eq. (28) implies that for $H = 1$ it is possible to have an efficiency as low as 50 per cent, but actually it can be shown that this is not the case, and that for H even somewhat smaller than 1, an efficiency of 66-2/3 per cent can always be obtained. For most applications this efficiency will be high compared to other inefficiencies in the system; thus Shannon-Fano coding of individual error terms solves problem for $H \geq 1$.

Coding Rare and Random Events—The next category that must be considered is message term distributions having an entropy $H < 1$. For this kind of message, Shannon-Fano coding of single message terms will not be useful, since each message term cannot be coded with less than one output binary digit, giving an $H_1 \geq 1$, which will be highly inefficient for small H. Indeed, it is clear that no method of coding single message terms into binary digits can be efficient. It is necessary to code a large number of terms at one time to obtain an output of much less than one binary digit per symbol. To avoid a large codebook memory, the coding must be non-nmemonic. That this is possible is due to the fact that a low-entropy discrete distribution must have one high-probability term. Indeed, even for $H = \frac{1}{2}$, the largest probability must be ≥ 0.89.[25] Low-entropy messages with uncorrelated message terms will then always consist of long runs of a single high-probability symbol, interrupted by occasional low-probability symbols. We will first consider coding methods for the simplest process of this type, in which there are only two symbols, and will then show that the general low-entropy distribution may be coded efficiently by a combination of these methods and Shannon-Fano coding.

Consider a sequence of zeros and ones. We assume that the successive values of the sequence are statistically independent. Let the probability of a zero be p, and the probability of a one, $1 - p = \epsilon$. We will define a run of k zeros as a segment of $100 \cdots 00$ of the sequence consisting of a one, which indicates the start of the run, followed by k zeros and a second one, which terminates the run. The second one is not counted as a part of the run, since it will be counted as the first symbol of the next run. Note that by this definition a succession of j ones is counted as $j - 1$ successive runs of zeros, each of length zero, followed by a run of zeros of some greater length.

To find the distribution of runs of zeros, we note that, given that a segment of the sequence *is* a run, we know that it starts with a one, and the probability that it has just k zeros is the probability that it has k zeros followed by a one. The distribution function for runs of zeros is thus

$$S(k) = (1 - p)p^k. \tag{29}$$

We may check that this is a reasonable probability distribution:

$$\sum_{k=0}^{\infty} S(k) = (1 - p) \sum_{k=0}^{\infty} p^k = 1$$

and find the mean number of zeros per run:

$$\sum_{k=0}^{\infty} kS(k) = (1 - p) \sum_{k=0} kp^k = p/(1 - p). \tag{30}$$

To code the sequence we describe each run by giving a binary number that indicates the number of zeros in the

[25] A two-term distribution with one probability $P_o = 0.89$ has an entropy $H = \frac{1}{2}$. Any distribution with more than two terms and with the largest probability $P_o < 0.89$ will have more entropy than this, by averaging Theorem II, Part I, since it can be obtained by an averaging operation on the two-term distribution. Therefore, if any distribution is to have $H = \frac{1}{2}$, it must have $P_o \geq 0.89$.

run.[26] The code number for a run of k zeros is obtained by omitting the first digit of the binary number $k + 1$.[27] The code for the sequence

$$1000000001000000100000001000001000000000$$

is thus the sequence

$$,001,11,000,10,010. \tag{31}$$

The coded version is shorter than the uncoded version, and this discrepancy will evidently increase with the average run length. And the difference in length would be even greater if it were not for the commas in the coded sequence. This is not a trivial matter—the commas, or some equivalent for them, are necessary, since we must know how the sequence of zeros and ones in the coded output is split up into code numbers describing individual run lengths. Because of the commas, the output is in a ternary, not a binary code, and each comma counts as one ternary output digit.

To find the average number of coded output digits for a run of zeros, note that it takes one comma for each run, no additional digits for a run of length zero, one additional digit for a run of length one or two; in general

$$C(k) = n + 1 \tag{32}$$

digits for a run of length k, where

$$\sum_{j=0}^{n-1} 2^j \leq k \leq \sum_{j=1}^{n} 2^j.$$

Now

$$\sum_{j=0}^{n-1} 2^j = 2^n - 1; \qquad \sum_{j=1}^{n} 2^j = 2^{n+1} - 2. \tag{33}$$

Using (29), (32), and (33) we can find $\overline{C}$, the average number of coded output digits per run,

$$\begin{aligned}\overline{C} &= \sum_{k=0}^{\infty} C(k)S(k)\\ &= 1 + [(1-p)/p]\Big[0p + 1\sum_{2}^{3} + \cdots\\ &\qquad\qquad + n\sum_{2^n}^{2^{n+1}-1} p^k + \cdots\Big]\\ &= 1 + [(1-p)/p][0/(1-p) + p^2/(1-p) + \cdots\\ &\qquad\qquad + p^{2^n}/(1-p) + \cdots]\end{aligned}$$

$$\overline{C} = 1 + (1/p)\sum_{n=1}^{\infty} p^{2^n}. \tag{34}$$

$\overline{C}$ is the number of ternary output digits per run. The equivalent number of binary digits per run is

$$\overline{C}\log_2 3. \tag{35}$$

From (30) the average number of zeros per run is $p/(1-p)$. There is a one at the start of each run, so the average number of input binary digits per run is

$$1 + p/(1-p) = 1/(1-p) \tag{36}$$

and the entropy per symbol of the input is, by definition,

$$p\log_2(1/p) + (1-p)\log_2[1/(1-p)]. \tag{37}$$

Thus the input entropy per run is the product of (36) and (37); and dividing this by (35) gives the efficiency or relative entropy R of the coding process

$$R = \frac{p\log_2(1/p) + (1-p)\log_2[1/(1-p)]}{(1-p)\overline{C}\log_2 3}. \tag{38}$$

This coding process is the second of a family of similar processes. The mth such process uses m symbols to code the lengths of runs of zeros in an m-ary code, the $(m+1)$st symbol being reserved for the commas in the coded message (31). For $m = 1$, the coding process reproduces the input signal. For $m = 3$, the coded output may be pairs of binary symbols, with the pair 00 being used for the commas. Larger values of m yield codes that are more efficient for very small $\epsilon = 1 - p$. Using the summation formulas

$$\sum_{j=0}^{n-1} m^j = (m^n - 1)/(m-1);$$

$$\sum_{j=1}^{n} m^j = (m^{n+1} - 1)/(m-1) - 1$$

in place of (32), a derivation like (33) gives the relations

$$\overline{C}(m, p) = 1 + (1/q)\sum_{n=1}^{\infty} q^{m^n}, \text{ with } q = p^{1/(m-1)} \tag{39}$$

$$R(m, p) = \frac{p\log_2(1/p) + (1-p)\log_2 1/(1-p)}{(1-p)\overline{C}(m,p)\log_2(m+1)} \tag{40}$$

of which (34) and (38) are the special cases $m = 2$. It is shown in the Appendix that

$$\lim_{p\to 1} R(m, p) = \log m/\log(m+1). \tag{41}$$

Therefore an m may be chosen to give as high an asymptotic efficiency as may be desired. As in the case of $m = 3$, some of the higher m values may conveniently be obtained by using combinations of binary or ternary digits to make up the necessary $m + 1$ message symbols: $m = 7$ is the next highest value realizable directly in binary digits.[28]

[26] Shannon (Shannon and Weaver, *op. cit.*, p. 33) discusses the problem of coding rare and random events as an example for discrete coding. He suggests sending a special code, such as a sequence of zeros, for the rare event, and using as a run-length code the number of zeros in the run, expressed in a modified binary number system that skips all numbers in which the special sequence occurs. He remarks that the procedure is asymptotically ideal for p near 1, i.e., as the rare event becomes rarer, providing that the length of the special sequence is properly adjusted. The modification used here is simpler to deal with analytically, and simpler to realize physically in a coding circuit. It also has the property of asymptotic ideality, as do a number of variants.

[27] The first digit on the left of a binary number is always a one, and supplies only positional information. This information is supplied by the comma in the sequence (31). It is necessary to add one to the number before removing the first digit in order to provide a representation for a run of length zero. In this code, such a run is indicated by two successive commas.

[28] m-ary digits with $m \neq 2^n - 1$ may, of course, be recoded into binary digits. This takes an amount of codebook memory and gives an efficiency of translation, which are as usual inversely related.

In Fig. 4, $R(m, p)$ is plotted against the logarithm to the base two of the average run length, $\log_2 [1/(1-p)]$, for m values of 1, 2, and 3. Each curve has a broad maximum, which moves to the right as m increases, the maximum being located approximately at the point where $\bar{C}(m, p) = m + 1$. This is to be expected, since at this point the frequency of the special symbol that indicates the start of a run will be equal to the average frequency of the other symbols. From the maximum, each curve approaches its asymptotic efficiency from above. The

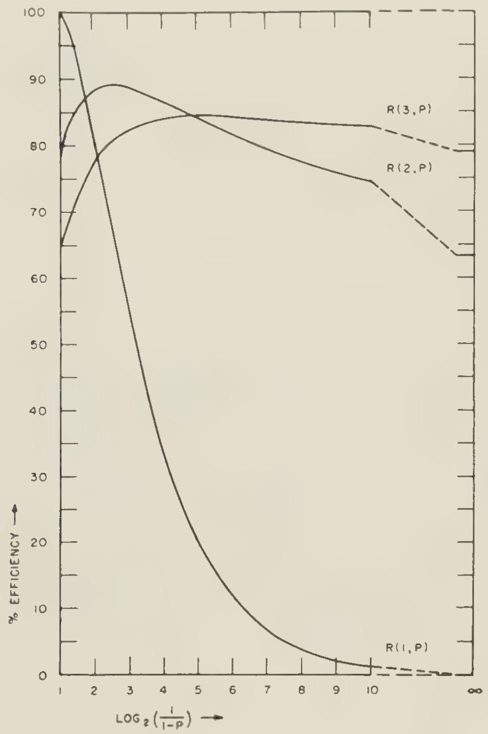

Fig. 4—Efficiency of non-nmemonic coding.

expression (41) for the asymptotic efficiency is the obvious statement that for p very near 1, the $(m+1)$st symbol which indicates the start of a run contributes very little information, since the runs are long and the symbol occurs very seldom. Note that for any value of p, one of the three codes shown will have an efficiency $R \geq 79$ per cent.

Case II: $H < 1$

Returning to the general low-entropy distribution, let the message terms be so labelled that M_0 is the term with high probability, and the rare terms are M_k, $1 \leq k \leq n$. Then define

$$\epsilon = \sum_{i=1}^{n} M_i$$

$$P_k = M_k/\epsilon.$$

The message may then be divided into two messages: one is a two-state process, with probabilities $p = 1 - \epsilon$ and ϵ, and the other is an n-state process with probability distribution P_k. The two-state process may be coded by the method described above, using m symbols to code lengths of runs of the common event and an $(m+1)$st symbol to indicate that a rare event has occurred, ending one run and starting the next. "The" rare event here is actually a signal that one of the low-probability symbols has occurred, and a term of a Shannon-Fano code for the probability distribution P_k follows the special symbol, to indicate which term of this distribution ended the run. Note that the Shannon-Fano coding need not be in the m-ary system used for the run-length codes. If an m value of 1 or 3 (or any other $m = 2^i - 1$) is used, and the $m+1$ symbols used are actually combinations of binary digits, then the Shannon-Fano code may be in ordinary binary notation. Since in Shannon-Fano coding the end of each coded term can be located by inspection, there will be no possibility of confusion.

The entropy of any low-entropy process is divisible into two terms that represent the two processes given above. We have

$$\begin{aligned} H &= -\sum_{i=0}^{n} M_i \log M_i \\ &= -M_0 \log M_0 - \sum_{i=1}^{n} (\epsilon P_i) \log (\epsilon P_i) \\ &= -M_0 \log M_0 - \epsilon \log \epsilon - \epsilon \sum_{i=1}^{n} P_i \log P_i \\ &= -(1-\epsilon) \log (1-\epsilon) - \epsilon \log \epsilon - \sum_{i=1}^{n} P_i \log P_i \\ &= H_2 + \epsilon H_n, \end{aligned} \tag{42}$$

where H_2 is the entropy of the two symbol process and H_n is the entropy of the distribution P_k. From (28) we have for H_n', the average number of binary digits per symbol in the Shannon-Fano code of the n-state process,

$$H_n' \leq H_n + 1. \tag{43}$$

For the two-state process, we have by the definition of R in Part II,

$$H_2' = H_2/R(m, p).$$

Using these two bounds and (42), the over-all efficiency of the combined coding is thus

$$R' = \frac{H_2 + \epsilon H_n}{H_2' + \epsilon H_n'} \geq \frac{H_2 + \epsilon H_n}{H_2/R(m, p) + \epsilon(1 + H_n)}, \tag{44}$$

and finally, since $R(m, p) < 1$ and all the terms are positive,

$$R' \geq \frac{H_2}{H_2/R(m, p) + \epsilon}. \tag{45}$$

Using only $m = 1$ and $m = 3$, taking the higher efficiency of the two,[29] and noting that H_2 is determined by ϵ,

$$H_2 = -\epsilon \log_2 \epsilon - (1-\epsilon) \log_2 (1-\epsilon), \tag{46}$$

[29] We are restricted to the use of a coding scheme that gives binary digits directly in evaluating (44) and (45), for the expression (43) is valid only for coding into binary digits. In the general case, coding into m-ary digits, the equivalent relation is $H_n' \leq H_n + \log_2 m$ bits.

we have that $R' = 66.7$ per cent at $\epsilon = \frac{1}{2}$. For smaller ϵ, R' never goes below about 65 per cent, and as $\epsilon \to 0$ $(p \to 1)$, R' approaches the asymptotic efficiency of the two-state coding scheme. This may be seen by noting that in the limit of small ϵ, (46) becomes[26]

$$H_2 \doteq \epsilon \log_2 (e/\epsilon). \tag{47}$$

Substituting this in (45),

$$R' \geq \frac{\log_2 (e/\epsilon) R(m, p)}{\log_2 (e/\epsilon) + R(m, p)}$$

and

$$\lim_{\epsilon \to 0} R' = \lim_{\epsilon \to 0} R(m, p) = \log m / \log (m + 1), \tag{48}$$

where the right side of (48) is given by (41). This is an asymptotic efficiency of 79 per cent for $m = 3$, and can, of course, be made as high as is desired by choice of sufficiently large m. For $\epsilon = \frac{1}{2}$, the entropy per symbol of the process $H_2 \geq 1$, by an argument like that in footnote 5, so we have a coding method with an efficiency $R' \geq 65$ per cent for all messages with successive message terms independently selected and with entropy $H \leq 1$.

Appendix

Asymptotic Efficiency of Run-Length Coding

Expressions for $\overline{C}(m, p)$ and $R(m, p)$ for p near one may be found by means of the exponential integral

$$EI(x) = \int_x^\infty (e^{-x}/x)\, dx.$$

First, for q less than one, q^{m^x} is a monotonic decreasing function of x, so we have the upper and lower bounds

$$U = \int_0^\infty q^{m^x}\, dx > \sum_{n=1}^{\infty} q^{m^n} > \int_1^\infty q^{m^x}\, dx = L. \tag{49}$$

Making the substitution $z = m^x \log(1/q)$, using natural logarithms, gives

$$U = (1/\log m) EI[\log (1/q)]$$
$$L = (1/\log m) EI[m \log (1/q)]. \tag{50}$$

Now from (39), for p near one,

$$q = p^{1/(m-1)} = (1 - \epsilon)^{1/(m-1)} \doteq 1 - (1 - p)/(m - 1).$$

Using this in (50) gives:

$$U = (1/\log m) EI[(1 - p)/(m - 1)]$$
$$L = (1/\log m) EI[m(1 - p)/(m - 1)].$$

Using the series expansion of EI and keeping the first two terms,[30]

$$U = \frac{-\gamma + \log (m - 1) + \log [1/(1 - p)]}{\log m}$$
$$L = U - 1,$$

where γ is Euler's constant, $\gamma = 0.5772 \cdots$.

From (39) and (50) we thus have, for p near one,

$$U + 1 > \overline{C}(m, p) > L + 1 = U. \tag{51}$$

From (36) and (37) we have the input entropy per run, converting to natural logarithms:

$$H/(1 - p) = 1/(1 - p)[(1 - p) \log 1/(1 - p) + p \log (1/p)].$$

For p near one, this becomes

$$H/(1 - p) \doteq \log [e/(1 - p)].$$

Recalling that each output digit is $(m + 1)$-ary, and is equivalent to $\log(m + 1)$ natural units, we have, for the efficiency $R(m, p)$ for p near one,

$$\frac{\log [e/(1 - p)]}{U \log (m + 1)} > R(m, p) > \frac{\log [e/(1 - p)]}{(U + 1) \log (m + 1)}.$$

Taking the limit gives the same result for each of the bounds so

$$\lim_{p \to 1} R(m, p) = \log m / \log (m + 1)$$

which is (41), the relation to be proved.

Acknowledgment

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Some of this material was presented by the author at the 1952 I.R.E. National Convention. At the same session papers were given by Oliver, Kretzmer, and Harrison, on television problems.[31] In particular, the paper by Harrison discusses *linear* predictive coding, but it makes use of a power, rather than an information, criterion for prediction. A note by the author[32] is also relevant to the linear predictive coding problem.

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The Linear, Input-Controlled, Variable-Pass Network*

B. E. KEISER†

Summary—**This paper describes the study and development of a linear, variable-pass network system which is controlled by the Fano short-time autocorrelation function of the input. Given an input function, the message, whose short-time power spectrum varies in an unpredictable manner with time, and to which there has been added a different function, the disturbance, whose short-time power spectrum is either time-invariant or varies in a completely known manner, a linear, input-controlled, variable-pass network can be specified which minimizes the mean-square error between the message input and the total output, taking network delay into account. Methods for mathematical computation of the mean-square error have been devised.**

The linear, input-controlled, variable-pass network has been found to have a lower mean-square error than that attainable with an optimum-mean-square, linear, fixed, selective network, for certain types of input messages.

Introduction

Recent years have witnessed the appearance of an extensive amount of literature on the theory of variable-pass networks.[1,2] Very little research has been done, however, on the capabilities of such networks in improving the transmission of information. This paper shows how a variable-pass network may be superior to a fixed, selective network in the reduction of the disturbance which often accompanies a desired message.

Let the desired message, expressed as a function of time, be denoted by $f_{i1}(t)$. Suppose that a disturbance, $f_{i2}(t)$, has been added unavoidably to the message. Let the sum, $f_{i1}(t) + f_{i2}(t)$, be denoted by $f_i(t)$. The function, $f_i(t)$, is regarded as the input to a selective network whose purpose is to operate upon $f_i(t)$ in such a manner that the network output, call it $f_0(t)$, is as nearly like the desired message, $f_{i1}(t)$, as is possible.

In terms of the frequency domain, a reduction in the bandwidth of the selective network generally reduces the amount of disturbance transmitted by it.[3] Too severe a reduction in bandwidth, however, generally produces intolerable distortion of the message. Therefore, one must decide what type of performance characterizes an optimum system for the problem at hand.

One criterion that has been discussed extensively in the literature,[4] and is relatively easy to deal with mathematically, is the minimization of the mean-square error, E_m. By definition,

$$E_m = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} [f_0(t) - f_{i1}(t)]^2 \, dt \tag{1}$$

is the mean-square error of $f_{i1}(t)$ and $f_0(t)$. If "a" denotes the delay encountered by $f_{i1}(t)$ in being transmitted through a selective network, an alteration in (1) is required in order that erroneous results not be obtained. The mean-square error of $f_{i1}(t-a)$ and $f_0(t)$ is

$$E_m^{(a)} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} [f_0(t) - f_{i1}(t-a)]^2 \, dt. \tag{2}$$

Practically speaking, the limit in (1) and (2) is taken as a value of T sufficiently large that essentially no change in the results would be obtained by taking a larger value.

The principal objective of optimum-mean-square, selective network theory is the minimization of expression (2). Ordinarily the apparent power density spectrum, $\Phi(\omega)$, or the autocorrelation function, $\phi(\tau)$, of the input message and disturbance are used to specify a time-invariant system. However, optimum-mean-square, selective network theory may be extended to the case of variable-pass devices, as will be shown in this paper.

The purposes of the investigation reported in this paper were:

1. A specification of the type of input functions which are capable of being processed more suitably by a variable-pass device than by a fixed, selective device.
2. The determination of the response of the variable-pass device which minimizes the mean-square error between message input and total output, provided the device is controlled by the Fano short-time autocorrelation function of the input.
3. A comparison of the mean-square error performance of an optimum, variable-pass device with that of an optimum, fixed, selective device designed to process the same input functions.

Functions with Time-Varying Spectra

The conventional definitions of the apparent power density spectrum, $\Phi(\omega)$, and the autocorrelation function, $\phi(\tau)$, require that average values involving the time function, $f(t)$, be taken over all values of time t. If $f(t)$ is not periodic, but if

$$0 < \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} [f(t)]^2 \, dt < \infty,$$

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[1] J. R. Carson and T. C. Fry, "Variable frequency electric circuit theory with application to the theory of frequency modulation," *Bell Sys. Tech. Jour.*, vol. 16, pp. 513–540; October, 1937.

[2] L. A. Zadeh, "Frequency analysis of variable networks," Proc. IRE, vol. 38, pp. 291–299; March, 1950.

[3] J. R. Carson, "Selective circuits and static interference," *Trans. AIEE*, vol. 43, pp. 789–796; 1924.

[4] N. Wiener, "The Extrapolation, Interpolation, and Smoothing of Stationary Time Series," The Technology Press, John Wiley and Sons, Inc., New York, N. Y., 1949.

then, by definition,[5]

$$\Phi(\omega) = \frac{1}{4\pi} \lim_{T\to\infty} \frac{1}{T} \left| \int_{-T}^{T} f(t)e^{-j\omega t}\, dt \right|^2 dt \qquad (3)$$

and

$$\phi(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)f(t-\tau)\, dt. \qquad (4)$$

The nature of $f(t)$ over an interval of t small compared with the entire interval is ignored completely when such definitions are used. A given $f(t)$ may possess markedly different properties over different ranges of time t as, for example, is evidenced by the fact that spectrograms presenting the frequency spectrum as a function of time have become very useful tools in the study of speech waves. A group of investigators[6] at the Bell Telephone Laboratories working on "visible speech" has found that such spectrograms can be read by trained people and therefore must contain much pertinent linguistic information regarding the speech wave from which they are obtained.

In order for a spectrum or correlation function to vary with time, more weight must be given to values of the function occuring at some times than at others. Although there are many ways in which short-time correlation functions and power spectra might be defined, Fano's definitions[7] appear to be the only ones that can be handled with ease mathematically and experimentally. The short-time autocorrelation function, $\phi_t(\tau, \alpha)$, is defined as

$$\phi_t(\tau, \alpha) = 2\alpha \int_{-\infty}^{t} f(x)f(x-\tau)e^{-2\alpha(t-x)}\, dx. \qquad (5)$$

The following discussion characterizes the type of input functions for which, in some cases, a variable-pass network proves superior to a fixed, selective network. Let a function with a time-varying frequency spectrum denote one whose short-time power spectrum is more sharply defined than is its long-time power spectrum at some or all times. The words "more sharply defined" should be given the following physical interpretation. Given two time functions whose frequency spectra have been determined on either a short- or a long-time basis, let the areas under each spectrum curve be made equal by a normalization process. The spectrum which then has the greater maximum ordinate will be referred to as being "more sharply defined." In general, the more sharply defined the message spectrum is, the lower the mean-square error can be made.

If a spectrum is more sharply defined on a short-time basis than on a long-time basis, a selective network operating on a short-time basis and controlled continuously by the present and immediately past statistical properties of the input is capable of separating a desired message from a disturbance more completely (on a mean-square error minimization basis) than would be possible with a fixed network. This is true since the fixed network necessarily must operate on a long-time basis. This requires, of course, that the message or disturbance, or both, be functions with time-varying spectra. The rate at which the spectral properties of the input vary governs the choice of the fixed weighting parameter α.

In general, functions with time-varying spectra have two types of variations, one of which characterizes rapid fluctuations of the functions, the other, slow spectral changes. A radio frequency carrier, phase or frequency modulated at a variable audio frequency rate, is an example of a function with one type of time-varying spectrum, that of time-varying frequency. Another type of function with a time-varying spectrum is encountered in the case of one with time-varying bandwidth. The fact that much phonograph music is of this type led Scott to design a variable-bandwidth amplifier,[8] bandwidth being controlled by reactance tubes which act in accordance with the presence or absence of certain frequency components in the input. This device, known as a dynamic noise suppressor, really is an input-controlled, variable-pass network, although it was not designed for the purpose of minimizing the mean-square error or for making optimum use of spectral properties of the input through correlation techniques.

The purpose of using Fano's short-time correlation function for controlling the variable-pass network is to detect slow spectral variations of the input message, $f_{i1}(t)$, which is assumed to be a function with a time-varying spectrum. Once the slow spectral variations have been detected, the transmission characteristics of the variable-pass network can be controlled accordingly.

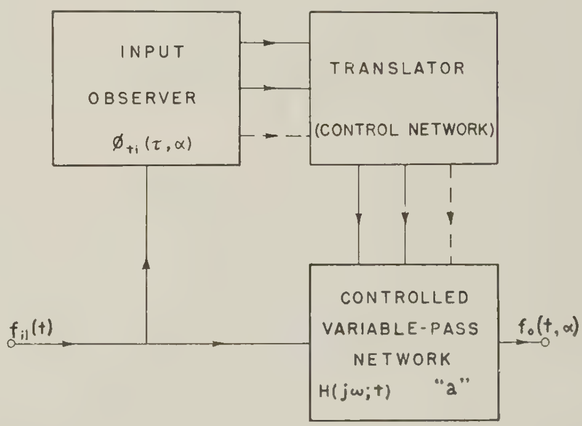

Fig. 1—Essential features of an input-controlled, variable-pass network.

Essential Features of an Input-Controlled, Variable-Pass Network

Fig. 1 is a block diagram showing the essential features of an input-controlled, variable-pass network. Its three chief components are shown there.

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[6] R. R. Riesz and L. Schott, "Visible speech cathode-ray translator," *Jour. Acous. Soc. Amer.*, vol. 18, pp. 50–61; January, 1946.

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The "input observer" is a Fano short-time autocorrelator; it obtains $\phi_{ti}(\tau, \alpha)$, the short-time autocorrelation function of $f_i(t)$. The "translator," or control network, is the device which translates $\phi_{ti}(\tau, \alpha)$ into control functions for the variable-pass network. In practice, the "translator" may consist of several time-invariant amplifiers with specified nonlinear amplitude characteristics. The reason for the nonlinearity is that, in general, there is no guarantee that the control functions for the variable-pass network itself will be linear functions of $\phi_{ti}(\tau, \alpha)$.

The transmission function of the variable-pass network is denoted by $H(j\omega; t)$. The required manner of variation of $H(j\omega; t)$ with $\phi_{ti}(\tau, \alpha)$ for mean-square error minimization is the subject of a later section of this paper.

Necessary Restrictions on the Input Message and Input Disturbance Functions

Now that a general picture of the operation of an input-controlled, variable-pass network has been obtained, the next step in its theoretical development is a specification of the restrictions which must be placed on the input message and the input disturbance in order to obtain time-varying control functions for the variable network. Although the input message is referred to as $f_{i1}(t)$ and the input disturbance as $f_{i2}(t)$, the choice of subscripts 1 and 2 is entirely arbitrary, and in the material which follows, the roles of the two, as well as the restrictions placed upon them, may be interchanged entirely if desired. In other words, the variable-pass network may be controlled on the basis of time variations in the short-time autocorrelation function of the disturbance rather than that of the message.

Let the assumption be made that $f_{i1}(t)$ is a function whose spectrum is either time-invariant, or one whose spectral variations are completely known for all values of time t. Furthermore, $f_{i1}(t)$ and $f_{i2}(t)$ are assumed to be uncorrelated with one another on a long-time basis, and essentially uncorrelated also on a short-time basis (specified by α). Then

$$\phi_{ti}(\tau, \alpha) = \phi_{ti1}(\tau, \alpha) + \phi_{ti2}(\tau, \alpha). \qquad (6)$$

Eq. (6) permits $\phi_{ti1}(\tau, \alpha)$ to be ascertained from an observation of $\phi_{ti}(\tau, \alpha)$.

An example of a case where $\phi_{ti2}(\tau, \alpha)$ is essentially time-invariant is obtained by letting $f_{i2}(t)$ be very wide-band random noise having no detectable spectral variations. For an example of a $\phi_{ti2}(\tau, \alpha)$ which corresponds to an $f_{i2}(t)$ whose spectral variations are "completely known," consider the following case. Suppose that in the frequency domain the function $f_{i1}(t)$ occupies region A only, but that function $f_{i2}(t)$ occupies both region A and region B, where A and B are mutually exclusive regions. If the spectral variations of $f_{i2}(t)$ are such that they can be determined completely by an observation that is limited to region B, then the results of this observation in region B can be utilized in the autocorrelation and message separation process being carried out in region A. In terms of (6), since $\phi_{ti}(\tau, \alpha)$ is observed directly in region A at each value of time t, and since $\phi_{ti2}(\tau, \alpha)$ is assumed to be known for all values of t from its behavior in region B, the present value of $\phi_{ti1}(\tau, \alpha)$ can be obtained at each instant of time.

The requirement that $f_{i1}(t)$ and $f_{i2}(t)$ be "essentially uncorrelated" on a short-time basis specified by α can be shown to be satisfied for all practical purposes provided $f_{i1}(t)$ and $f_{i2}(t)$ are not sine waves separated by less than α/π cycles per second, or provided they are not random functions whose spectra are concentrated in a common bandwidth of the order of magnitude of α/π. Zero long-time cross-correlation, of course, also is required.

A Theorem Relating the Average Value of a Weighted Function to the Average of the Function

In comparing an input-controlled, variable-pass network with a fixed, selective network, one generally is interested in the long-time behavior of each. A theorem of considerable use in relating short-time properties of functions to their long-time properties is the following:

Theorem I-A: if $F(t)$ is a bounded, continuous, single-valued function of t, and if its derivative, $F'(t)$, is continuous, then

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} F(t)\, dt = \lim_{T\to\infty} \frac{\alpha}{T} \int_{-T}^{T} \int_{-\infty}^{t} F(x) e^{-2\alpha(t-x)}\, dx\, dt$$
$$0 < \alpha < \infty, \alpha \text{ real}. \qquad (7)$$

This theorem may be proved by use of integration by parts and by application of the mean-value theorem for integrals. The theorem may be used to obtain the relationship between the short-time cross-correlation of two random (but bounded) functions. (If these functions are voltages or currents, they are said to be of the finite power type.)

Let $F(t) = f_A(t)\, f_B(t - \tau)$. Application of (7) yields

$$\phi_{A,B}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \phi_{tA,B}(\tau, \alpha)\, dt. \qquad (8)$$

If $f_A(t) \equiv f_B(t)$, then the expression obtained is a relationship between short- and long-time autocorrelation functions, and the subscripts A and B may be dropped. Thus

$$\phi(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \phi_t(\tau, \alpha)\, dt. \qquad (9)$$

If $F(t)$ is such that

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} F(t)\, dt = 0, \qquad (10)$$

Theorem I-A still is valid, but is of little direct use. However, a relationship similar to Theorem I-A may be proved.

Theorem I-B: if $F(t)$ is a single-valued, bounded function for which

$$\int_{-\infty}^{\infty} F(t)\, dt < \infty,$$

then

$$\int_{-\infty}^{\infty} F(t)\,dt = 2\alpha \int_{-\infty}^{\infty} \int_{-\infty}^{t} F(x)e^{-2\alpha(t-x)}\,dx\,dt$$

$$0 < \alpha < \infty, \alpha \text{ real}. \quad (11)$$

Letting $F(t) = f_A(t)\,f_B(t-\tau)$ yields

$$\phi_{eA,B}(\tau) = \int_{-\infty}^{\infty} \phi_{tA,B}(\tau, \alpha)\,dt \quad (12)$$

where, by definition,

$$\phi_{eA,B}(\tau) = \int_{-\infty}^{\infty} f_A(t) f_B(t-\tau)\,dt \quad (13)$$

is the long-time cross-correlation of two functions having finite energy. If $f_A(t) \equiv f_B(t)$, then the expression obtained is a relationship between short- and long-time autocorrelation functions. Thus

$$\phi_e(\tau) = \int_{-\infty}^{\infty} \phi_t(\tau, \alpha)\,dt. \quad (14)$$

Mean-Square Error in Terms of Correlation Functions of the Input and Output

The mean-square error of a variable-pass network depends not only upon the transmission function, $H(j\omega; t)$, as in fixed, selective network theory, but also upon the way in which $H(j\omega; t)$ varies with time, and thus upon $\phi_{ti}(\tau, \alpha)$. Let $E_m^{(a)}(\alpha)$ denote the mean-square error of a variable-pass network, provided the transmission function is the optimum one for a given network delay, "a," and a given input, and provided this optimum transmission function is controlled in accordance with the short-time autocorrelation function of the input. Once $E_m^{(a)}(\alpha)$ has been found, there still remains the problem of finding that value of α for which $E_m^{(a)}(\alpha)$ is a minimum. Since the output of an input-controlled, variable-pass network depends upon α, long-time cross-correlation and autocorrelation functions involving the output also can be expected to depend upon α. Expansion of (2), with $f_0(t)$ replaced by $f_0(t, \alpha)$ and $E_m^{(a)}$ replaced by $E_m^{(a)}(\alpha)$, application of the definition of the short-time correlation function, and utilization of Theorem I-A, lead, under the restrictions developed previously, to a proof that

$$E_m^{(a)}(\alpha) = \phi_{01}(0, \alpha) - 2\phi_{01,i1}(a, \alpha) + \phi_{02}(0, \alpha) + \phi_{i1}(0). \quad (15)$$

Eq. (15) was obtained on the assumption that $f_{i1}(t)$, and hence $f_0(t, \alpha)$, do not tend to zero as $t \to \pm\infty$. If either $f_{i1}(t)$ or $f_{i2}(t)$ or both are functions which do tend to zero as $t \to \pm\infty$, then instead of studying the mean-square error, the quantity which must be studied is

$$\int_{-\infty}^{\infty} [f_0(t, \alpha) - f_{i1}(t-a)]^2\,dt.$$

Wherever the mean-square error has been obtained, the quantity

$$\int_{-\infty}^{\infty} [f_0(t, \alpha) - f_{i1}(t-a)]^2\,dt$$

also can be found, provided the given functions satisfy Theorem I-B instead of Theorem I-A. (An example of a function of this type is a finite train of pulses.) A proof, similar to the one for (15), has been obtained that

$$\int_{-\infty}^{\infty} [f_0(t, \alpha) - f_{i1}(t-a)]^2\,dt$$

$$= \phi_{e01}(0, \alpha) - 2\phi_{e01,i1}(a, \alpha) + \phi_{e02}(0, \alpha) + \phi_{ei1}(0). \quad (16)$$

Eqs. (15) and (16) are similar to expressions which have been obtained in the literature for the mean-square error of fixed, selective networks. For fixed networks, however, no parameter α is involved. Eq. (15) has been verified experimentally.

By means of a theorem relating the short-time cross-correlation of two functions to their short-time autocorrelations, and by means of a theorem relating the short-time autocorrelation of the output of a variable-pass network to the short-time power spectrum of the input, it is possible to find the mean-square error of an input-controlled, variable-pass network in terms of short-time autocorrelation functions of the input only. The relationships obtained, however, are rather lengthy, and therefore are not presented here.

The Control of the Transmission Function, $H(j\omega; t)$

If the transmission function, $H(j\omega; t)$, could be controlled perfectly by values assumed by $\phi_{ti1}(\tau, \alpha)$ and $\phi_{ti2}(\tau, \alpha)$, the mean-square error of a variable-pass network would consist only of a message error and a disturbance error. However, when message and disturbance are present simultaneously at the input of an input-controlled, variable-pass network, there arises an additional error, which will be called "tracking error." When such an error is of a negligible order of magnitude, the following results, stated in Theorem II, can be obtained.

Theorem II: let $\Phi_{ti1}^{(a)}(\omega, \alpha)$ be the short-time power spectrum of a function with a time-varying spectrum, and let $\Phi_{ti2}^{(a)}(\omega, \alpha)$ be a short-time power spectrum which either is time-invariant, or whose time variations can be determined by a process independent of the presence of $f_{i1}(t)$. If two functions, $f_{i1}(t)$ and $f_{i2}(t)$, possess essentially zero cross-correlation for all τ and all t, let these functions be the input message and input disturbance, respectively, to a stable, variable-pass network with a transmission function, $H(j\omega; t)$. Let $H(j\omega; t)$ at each instant of time t be given the values, $H_0(j\omega; t)$, that an optimum, fixed, transmission function would be given for an input with conventional power spectrum functions, $\Phi_{ti1}^{(a)}(\omega, \alpha)$ and $\Phi_{ti2}^{(a)}(\omega, \alpha)$ at t. Then $E_m^{(a)}(\alpha)$, which is the sum of the message error and the disturbance error of the variable-pass network, assumes its minimum value for a given "a" and α. Instead of using $\Phi_{ti1}^{(a)}(\omega, \alpha)$ and $\Phi_{ti2}^{(a)}(\omega, \alpha)$, control on the basis of correlation functions may be used if, in place of the long-time correlation functions the quantities, $e^{-\alpha|\tau|}\,\phi_{ti1}^{(a)}(\tau, \alpha)$ and $e^{-\alpha|\tau|}\,\phi_{ti2}^{(a)}(\tau, \alpha)$, are used.

Results Obtained Experimentally

Fig. 2 shows the basic elements of the experimental equipment. The message, $f_{i1}(t)$, consisted of a random voltage of bandwidth 60 cycles per second and of center frequency between 2.5 and 3.0 kilocycles per second. The center frequency was a random function of time. Its values fluctuated in an unpredictable manner about 2.75 kilocycles per second. The disturbance was very wide-band random noise possessing no short-time spectral variations.

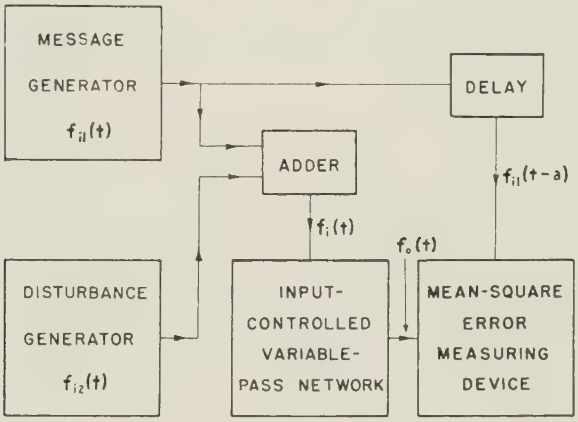

Fig. 2—Basic elements of experimental equipment.

For the particular combination of message and disturbance chosen, the variable-pass network showed improvements as listed in the following table:

TABLE I
Reduction of $[f_0(t) - f_{i1}(t-a)]$ Obtained Experimentally

Noise-to-Signal Power Ratio	No Network Inserted	Fixed Network	Variable-Pass Network
1.5	0 db.	2.1 db.	5.5 db.
1.0	0 db.	2.3 db.	6.0 db.
0.75	0 db.	2.6 db.	6.9 db.
0.50	0 db.	3.1 db.	8.1 db.
0.25	0 db.	3.5 db.	9.6 db.

The noise-to-signal power ratio was that measured in a 500 cycles per second bandwidth. For each noise-to-signal power ratio, the optimum-mean-square selective network possesses a different bandwidth. In the experimental example chosen, the variable-pass network is able to operate with a fixed bandwidth of about 100 cycles per second, while the corresponding fixed network would require a bandwidth of about 500 cycles per second. A 5-to-1 reduction in bandwidth should yield a decrease in noise power transmission of about 7.0 decibels. Table I shows that this value was approached at relatively low disturbance levels. The poorer improvements obtained at higher disturbance levels may be attributed to tracking error, arising chiefly from the fact that the short-time autocorrelator transmits a certain amount of noise. (The noise power transmitted by a Fano short-time autocorrelator is proportional roughly to α and also to the noise power level at its input.)

Although the example treated experimentally was one in which bandwidth was fixed (only center frequency was variable), much more complicated types of variation could be handled if necessary. The simplifications employed permitted a device to be constructed in which only two τ values of the short-time autocorrelation function were used in the control of the variable-pass network. More complicated spectral variations also might be handled by means of a short-time autocorrelation function other than Fano's. However, the mathematical theorems developed in this research are valid only when the Fano autocorrelation function of the network input is used as a basis of control.

Specification of the Parameter α

One question which may have arisen by this time is how short a short-time autocorrelation function should be used. No precise analytical answer can be obtained in the general case without a considerable amount of data. However, a simple answer in physical terms can be given which should be of value in many cases. Let $\xi(t)$ denote the varying property of the input message spectrum. Thus $\xi(t)$ may denote bandwidth, center frequency, or some special combination of these. Let a Fourier analysis of $\xi(t)$ be made, thus yielding the spectrum of the variation. Choose α such that the Fano correlator's low-pass network (bandwidth: α/π cycles per second) passes essentially all frequency components of the spectrum of the variation without undue attenuation. Aside from this requirement the bandwidth of the low-pass network should be kept as small as possible.

Conclusions

A sufficient condition that an input-controlled, variable-pass network be superior to a fixed, selective network is that the short-time spectrum of at least one of the two input functions, $f_{i1}(t)$ and $f_{i2}(t)$, be more sharply defined than its corresponding long-time spectrum, over some, but not all values of time. (By assumption, $f_{i1}(t)$ and $f_{i2}(t)$ possess essentially zero short-time cross-correlation at the value of α utilized.) A necessary condition that a variable-pass network be superior to a fixed, selective network is that at least one of its two input functions, $f_{i1}(t)$ and $f_{i2}(t)$, be a function with a time-varying spectrum over some, but not all, values of time.

The extent of the improvement attainable depends upon the spectral properties of the input message and disturbance, i.e., bandwidths on short- and long-time bases, as well as rates of variation. If neither of the two input functions have a time-varying spectrum, the variable-pass network degenerates into a fixed, selective network.

Mathematical Notation Used

E_m	mean-square error.
$E_m^{(a)}$	mean-square error of a fixed, selective network having a delay of "a" seconds.

$E_m^{(a)}(\alpha)$	mean-square error of a linear, input-controlled, variable-pass network having delay "a" and correlation weighting parameter α.
$H(j\omega; t)$	transmission function of a variable-pass network.
a	network delay in seconds.
$f(t)$	a function of time.
$f_i(t)$	total input to network.
$f_{i1}(t)$	input message.
$f_{i2}(t)$	input disturbance.
$f_0(t)$	total output of network.
$f_0(t, \alpha)$	total output of a variable-pass network controlled by a short-time autocorrelator having weighting parameter α.
$f_{01}(t, \alpha)$	message component of total output of variable-pass network.
$f_{02}(t, \alpha)$	disturbance component of total output of variable-pass network.
t	time in seconds.
$\Phi(\omega)$	long-time (conventional) apparent power density spectrum.
$\Phi_{ti1}(\omega, \alpha)$	Fano short-time power spectrum of $f_{i1}(t)$.
$\Phi_{ti1}^{(a)}(\omega, \alpha)$	Fano short-time power spectrum which $f_{i1}(t)$ possessed "a" seconds ago.
$\Phi_{ti2}(\omega, \alpha)$	Fano short-time power spectrum of $f_{i2}(t)$.
$\Phi_{ti2}^{(a)}(\omega, \alpha)$	Fano short-time power spectrum which $f_{i2}(t)$ possessed "a" seconds ago.
α	the parameter which governs the weight given to past values of a function in determining its short-time correlation function. It is equal, numerically, to the half-bandwidth of a (short-time) spectrum analyzer in radians per second.
ξ	a time-varying property of a short-time spectrum.
τ	the time by which one of the functions is delayed in a correlation process.
$\phi(\tau)$	long-time (conventional) autocorrelation of a function possessing a nonzero apparent power density spectrum.
$\phi_A(\tau)$	long-time (conventional) autocorrelation of $f_A(t)$, where $f_A(t)$ possesses a nonzero apparent power density spectrum.
$\phi_{A,B}(\tau)$	long-time (conventional) cross-correlation function of $f_A(t)$ and $f_B(t)$, functions which possess nonzero apparent power density spectra.
$\phi_e(\tau)$	long-time (conventional) autocorrelation of function with a finite energy density spectrum.
$\phi_{eA}(\tau)$	long-time (conventional) autocorrelation of a function, $f_A(t)$, possessing a finite energy density spectrum.
$\phi_{eA,B}(\tau)$	long-time (conventional) cross-correlation of $f_A(t)$ and $f_B(t)$, functions which possess finite energy density spectra.

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Spectral Power Density Functions in Pulse Time Modulation

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Summary—Spectral power density functions corresponding to various types of pulse shapes, and probability distribution functions arising in the study of pulse time modulation problems are computed. The results are presented in tabular form. The following cases are considered: PAM and PPM, for arbitrary pulse shape, PDM, for rectangular, Gaussian, and error-function pulse shapes, and SEM, for rectangular pulse shape.

Introduction

In analysis of the performance of systems composed of linear filters and having pulse train inputs, many problems consist of determining the behavior of such systems when the pulse train inputs have random fluctuations in pulse amplitudes, pulse positions, or pulse durations. In the setting up of these problems, it is convenient to have available standard types of idealized pulse forms, commonly used probability density functions, and spectral power density functions for idealized types of pulse time modulation. The chief purpose of this paper is to make available, in tabular form, a compilation of spectral power density functions for various pulse shapes, various probability density distributions, and various types of pulse time modulation that the authors have found to be useful tools in the analytical evaluation of systems' performances.

The tables do not exhaust all the possibilities, but are intended to cover many of the commonly used types for which solutions could be found in closed form. The derivation of the spectral power density functions for a few cases have appeared in the literature.[1-5] MacFarlane's analysis[1] served as basis for the derivation of the additional spectral power density functions included. A closely related study, using the correlation function approach, is given by Kretzmer.[2]

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[1] G. G. MacFarlane, "On the energy-spectrum of an almost periodic succession of pulses," Proc. I.R.E., vol. 37, pp. 1139–1143; October, 1949. *See also* discussion of this paper by T. S. George, Proc. I.R.E., vol. 38, pp. 1212–1213.

[2] E. R. Kretzmer, "Interference Characteristics of Pulse-Time Modulation," RLE Report No. 92, Mass. Inst. of Tech.; May 27, 1949.

[3] J. L. Lawson and G. E. Uhlenbeck, "Threshold Signals," Radiation Lab. Series, vol. 24, McGraw-Hill Publishing Co., New York, ch. 3, sec. 3.4, pp. 42–46; 1950.

[4] E. R. Kretzmer, "An application of auto-correlation analysis," *Jour. Math. Phys.*, vol. 29, pp. 179–190; October, 1950.

[5] Z. Jelonek, "Noise problems in communication," *Jour. IEE*, Part IIIA, vol. 94, pp. 533–545; 1947.

Summary of Formulas

The direct and inverse Fourier Transforms as used here are given by

$$G(\omega) = \int_{-\infty}^{\infty} f(t) \exp(-j\omega t)\, dt \tag{1a}$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) \exp(j\omega t)\, d\omega, \tag{1b}$$

where $f(t)$ and $G(\omega)$ are the time function and the spectrum, respectively, of a typical member of the pulse train and are given in Table I. (Tables I–VI follow page 41, and are printed consecutively.)

If $q(x)$ is the probability function of a random variable, x, the discrete and continuous spectral power density functions for pulse amplitude modulation, PAM, are

$$P_d(\omega) = \frac{2\pi}{T^2} \mid G(\omega) \mid^2 X^2 \delta(\omega - m\omega_p), \quad m = 0, \pm 1, \pm 2 \cdots \tag{2a}$$

$$P_c(\omega) = \frac{1}{T} \mid G(\omega) \mid^2 \overline{(x - X)^2}, \tag{2b}$$

where $P_d(\omega)$ represents the line spectrum of the periodic components, $P_c(\omega)$ represents the continuous power density per cycle per second, $\omega_p = 2\pi/T$, mean repetition frequency, in radians per second,

$$X = \int_{-\infty}^{\infty} xq(x)\, dx, \tag{3a}$$

$$\overline{(x - X)^2} = \int_{-\infty}^{\infty} (x - X)^2 q(x)\, dx = \int_{-\infty}^{\infty} x^2 q(x)\, dx - X^2. \tag{3b}$$

The discrete and continuous spectral power density functions for pulse position modulation, PPM, are

$$P_d(\omega) = \frac{2\pi}{T^2} \mid G(\omega) \mid^2 \mid X \mid^2 \delta(\omega - m\omega_p), \quad m = 0, \pm 1, \pm 2, \cdots \tag{4a}$$

$$P_c(\omega) = \frac{1}{T} \mid G(\omega) \mid^2 \overline{(x - X)^2}; \tag{4b}$$

where

$$X = \int_{-\infty}^{\infty} \exp(j\omega x) q(x)\, dx, \tag{5a}$$

$$\overline{(x - X)^2} = \int_{-\infty}^{\infty} \exp(j\omega x) \exp(-j\omega x) q(x)\, dx$$
$$- | X |^2 = 1 - | X |^2. \qquad (5b)$$

The discrete and continuous spectral power density functions for pulse duration modulation, PDM, are

$$P_d(\omega) = \frac{2\pi}{T^2} X^2 \delta(\omega - m\omega_p), \quad m = 0, \pm 1, \pm 2, \cdots \qquad (6a)$$

$$P_c(\omega) = \frac{1}{T} \overline{(x - X)^2}; \qquad (6b)$$

where

$$X = \int_{-\infty}^{\infty} G(\omega, x) q(x)\, dx, \qquad (7a)$$

$$\overline{(x - X)^2} = \int_{-\infty}^{\infty} G^2(\omega, x) q(x)\, dx - X^2. \qquad (7b)$$

The discrete and continuous spectral power density functions for single edge modulation, SEM, are

$$P_d(\omega) = \frac{2\pi}{T^2} | X |^2 \delta(\omega - m\omega_p), \; m = 0, \pm 1, \pm 2, \cdots \qquad (8a)$$

$$P_c(\omega) = \frac{1}{T} \overline{(x - X)^2}; \qquad (8b)$$

where X is given by (7a), and

$$\overline{(x - X)^2} = \int_{-\infty}^{\infty} | G(\omega, x) |^2 q(x)\, dx - | X |^2. \qquad (9)$$

In (3a), and (3b), x is the random variable defining pulse amplitudes. The $q(x)$ is defined in Table II, taking $\bar{A} = 1$. In (5a) and (5b), x is the random variable defining a pulse position relative to the unmodulated position of the preceding pulse, and the $q(x)$ is defined in Table II, taking $\bar{A} = T$. In (7a) and (7b), x is the random variable defining the pulse duration, and $q(x)$ is defined in Table II, taking $\bar{A} = \tau$. In (9), x is the random variable defining the displacement of either the leading edges or the trailing edges of the pulses from the positions occupied by the centers of unmodulated pulses. The $q(x)$ is defined in Table II, where for leading edge modulation, $\bar{A} = -\tau/2$, and for trailing edge modulation, $\bar{A} = \tau/2$.

The power contained in the real frequency band extending from ω_1 to ω_2 is given by

$$R(\omega_1, \omega_2) = \frac{1}{2\pi} \int_{\omega_1}^{\omega_2} [P_d(\omega) + P_c(\omega)]\, d\omega$$
$$+ \frac{1}{2\pi} \int_{-\omega_2}^{-\omega_1} [P_d(\omega) + P_c(\omega)]\, d\omega$$
$$= \frac{1}{\pi} \int_{\omega_1}^{\omega_2} [P_d(\omega) + P_c(\omega)]\, d\omega. \qquad (10)$$

Example

The discrete spectral power density function of a rectangular pulse train having a constant pulse amplitude, E, a constant pulse duration, τ, and a repetition period, T, is given by

$$P_d(\omega) = \frac{2\pi}{T^2} (E\tau)^2 \frac{\sin^2(\omega\tau/2)}{(\omega\tau/2)^2} \delta(\omega - m\omega_p),$$
$$m = 0, \pm 1, \pm 2, \text{etc.}$$

The average power in this pulse train is given by

$$R(-\infty, \infty) = \frac{1}{2\pi} \int_{-\infty}^{\infty} P_d(\omega)\, d\omega$$
$$= \frac{(E\tau)^2}{T^2} \int_{-\infty}^{\infty} \frac{\sin^2(\omega\tau/2)}{(\omega\tau/2)^2} \delta(\omega - m\omega_p)\, d\omega$$
$$= \frac{(E\tau)^2}{T^2} \sum_{-\infty}^{\infty} \frac{\sin^2(m\omega_p\tau/2)}{(m\omega_p\tau/2)^2} = E^2 \frac{\tau}{T}.$$

Description of Tables

Table I consists of five idealized pulse forms together with their Fourier transforms. Note that the pulse length in every case except that of the square pulse is defined as the time between half-amplitude points of the time functions. This lends a convenient symmetry to the Fourier transforms, and in no way lessens the generality of the results.

Table II consists of four probability density functions frequently encountered in systems' analyses.

Tables III to VI present the spectral power density functions for various random time modulations. The spectral power density functions consist of two parts. The discrete spectral power density function, $P_d(\omega)$, represented by the Dirac delta functions at the harmonics of the repetition frequency, is a measure of the power distribution in the periodic components of the pulse train. Similarly, the distribution of power continuously spread over the frequency domain is given by the continuous spectral power density function, $P_c(\omega)$. All six tables mentioned above appear on pages 42 through 46 immediately following this paper.

TABLE I—PULSE SHAPES AND TRANSFORMS

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\exp(j\omega t)\,d\omega, \qquad G(\omega) = \int_{-\infty}^{\infty} f(t)\exp(-j\omega t)\,dt$$

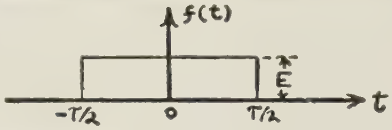
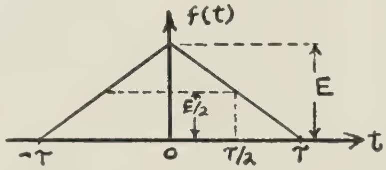
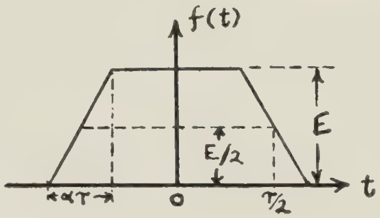
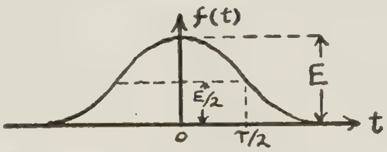
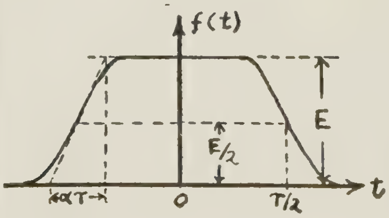
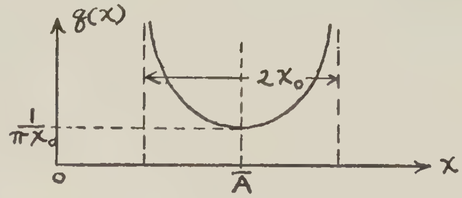
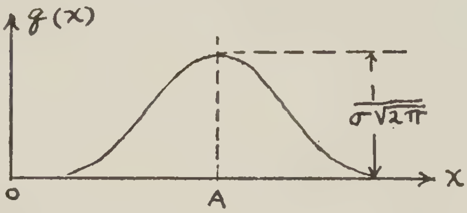
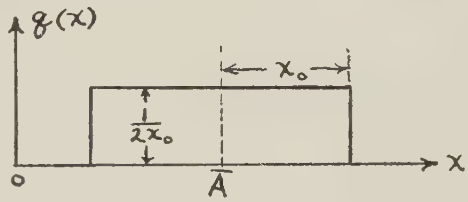
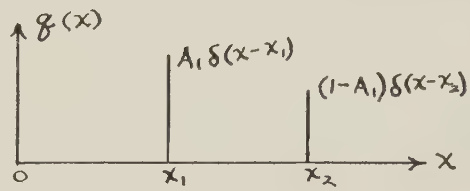
Type	Graph	$f(t)$	$G(\omega)$
Rectangular		$E, \quad \lvert t \rvert \le \tau/2$ $0, \quad \lvert t \rvert > \tau/2$	$E\tau \dfrac{\sin\left(\frac{\omega\tau}{2}\right)}{\left(\frac{\omega\tau}{2}\right)}$
Triangular		$\frac{E}{\tau}(t+\tau), \quad 0 \ge t \ge -\tau$ $-\frac{E}{\tau}(t-\tau), \quad 0 \le t \le \tau$ $0, \quad \lvert t \rvert > \tau$	$E\tau \dfrac{\sin^2\left(\frac{\omega\tau}{2}\right)}{\left(\frac{\omega\tau}{2}\right)^2}$
Trapezoidal		$\frac{E}{\alpha\tau}\left[t + \frac{\tau}{2}(1+\alpha)\right],$ $-\frac{\tau}{2}(1+\alpha) \le t \le -\frac{\tau}{2}(1-\alpha)$ $E, \quad \lvert t \rvert < \frac{\tau}{2}(1-\alpha)$ $-\frac{E}{\alpha\tau}\left[t - \frac{\tau}{2}(1+\alpha)\right],$ $\frac{\tau}{2}(1-\alpha) \le t \le \frac{\tau}{2}(1+\alpha)$ $0, \quad \lvert t \rvert > \frac{\tau}{2}(1+\alpha)$	$E\tau \dfrac{\sin\left(\frac{\omega\tau}{2}\right)}{\left(\frac{\omega\tau}{2}\right)}$ $\cdot \dfrac{\sin\left(\frac{\alpha\omega\tau}{2}\right)}{\left(\frac{\alpha\omega\tau}{2}\right)}$
Gaussian		$E\exp\left[-4(\ln 2)\frac{t^2}{\tau^2}\right], \quad -\infty < t < \infty$	$E\tau\sqrt{\frac{\pi}{4\ln 2}}$ $\cdot \exp\left(\frac{-\omega^2\tau^2}{16\ln 2}\right)$
Error function		$\frac{E}{2}\left[\operatorname{erf}\frac{\sqrt{\pi}}{\alpha}\left(\frac{t}{\tau}+\frac{1}{2}\right)\right.$ $\left.-\operatorname{erf}\frac{\sqrt{\pi}}{\alpha}\left(\frac{t}{\tau}-\frac{1}{2}\right)\right], \quad -\infty < t < \infty$ Note: $f(0) = E\operatorname{erf}\frac{\sqrt{\pi}}{2\alpha} \approx E$ $f\left(\frac{\tau}{2}\right) = \frac{E}{2}\operatorname{erf}\frac{\sqrt{\pi}}{2\alpha} \approx \frac{E}{2}$ $f'\left(-\frac{\tau}{2}\right) \approx \frac{E}{\alpha\tau}$	$E\tau \dfrac{\sin\left(\frac{\omega\tau}{2}\right)}{\left(\frac{\omega\tau}{2}\right)}$ $\cdot \exp\left(\frac{-\alpha^2\omega^2\tau^2}{4\pi}\right)$

TABLE II—PROBABILITY DENSITY FUNCTIONS, $q(x)$

Type	$q(x)$	Graph of $q(x)$	Mean $X = \int_{-\infty}^{\infty} x \cdot q(x)\, dx$	Mean Squared Deviation $\overline{(x-X)^2}$
Sinusoidal	$[\pi\sqrt{x_0^2-(x-\bar{A})^2}]^{-1}$, $\lvert x-\bar{A}\rvert < x_0$ $0,\ \lvert x-\bar{A}\rvert > x_0$		$\bar{A}$	$\frac{x_0^2}{2}$
Normal	$(\sigma\sqrt{2\pi})^{-1} \cdot \exp\left(\frac{-[x-\bar{A}]^2}{2\sigma^2}\right)$, $-\infty < x < \infty$		$\bar{A}$	σ^2
Flat	$\frac{1}{2x_0},\ \lvert x-\bar{A}\rvert \le x_0$ $0,\ \lvert x-\bar{A}\rvert > x_0$		$\bar{A}$	$\frac{x_0^2}{3}$
Double Spike	$A_1\delta(x-x_1)$ $+ (1-A_1)\delta(x-x_2)$		$A_1(x_1-x_2) + x_2$	$A_1(1-A_1) \cdot (x_1-x_2)^2$

Note: All $q(x)$ are normalized: $\int_{-\infty}^{\infty} q(x)\, dx = 1$

TABLE III—RANDOM PULSE AMPLITUDE MODULATION, (PAM)

Distribution Function	Discrete Spectral Power Density Function, $P_D(\omega)$	Continuous Spectral Power Density Function, $P_C(\omega)$
Sinusoidal	$\frac{2\pi}{T^2} \lvert G(\omega)\rvert^2 \delta(\omega - m\omega_p)$	$\frac{x_0^2}{2T} \lvert G(\omega)\rvert^2$
Normal	$\frac{2\pi}{T^2} \lvert G(\omega)\rvert^2 \delta(\omega - m\omega_p)$	$\frac{\sigma^2}{T} \lvert G(\omega)\rvert^2$
Flat	$\frac{2\pi}{T^2} \lvert G(\omega)\rvert^2 \delta(\omega - m\omega_p)$	$\frac{x_0^2}{3T} \lvert G(\omega)\rvert^2$
Double Spike	$\frac{2\pi}{T^2} [A_1(x_1-x_2)+x_2]^2 \lvert G(\omega)\rvert^2 \delta(\omega - m\omega_p)$	$\frac{A_1(1-A_1)(x_1-x_2)^2}{T} \lvert G(\omega)\rvert^2$

$$\omega_p = \frac{2\pi}{T}, \qquad m = 0, \pm 1, \pm 2, \cdots$$

Note: For the double spike case, $G(\omega)$ is taken from Table I with $E = 1$. For the other three cases, $q(x)$ as given in Table II, have mean values of 1.

TABLE IV—RANDOM PULSE POSITION MODULATION, (PPM)

Distribution Function	Discrete Spectral Power Density Function, $P_D(\omega)$	Continuous Spectral Power Density Function, $P_C(\omega)$
Sinusoidal	$\frac{2\pi}{T^2} J_0^2(\omega x_0) \mid G(\omega) \mid^2 \delta(\omega - m\omega_p)$	$\frac{\mid G(\omega) \mid^2}{T} [1 - J_0^2(\omega x_0)]$
Normal	$\frac{2\pi}{T^2} \exp(-\sigma^2\omega^2) \mid G(\omega) \mid^2 \delta(\omega - m\omega_p)$	$\frac{\mid G(\omega) \mid^2}{T} [1 - \exp(-\sigma^2\omega^2)]$
Flat	$\frac{2\pi}{T^2} \left[\frac{\sin(\omega x_0)}{\omega x_0}\right]^2 \mid G(\omega) \mid^2 \delta(\omega - m\omega_p)$	$\frac{\mid G(\omega) \mid^2}{T} \left[1 - \left\{\frac{\sin(\omega x_0)}{\omega x_0}\right\}^2\right]$
Double Spike	$\frac{2\pi}{T^2} \left[1 - 4A_1(1 - A_1) \sin^2\left(\frac{\omega x_1 - \omega x_2}{2}\right)\right] \cdot \mid G(\omega) \mid^2 \delta(\omega - m\omega_p)$	$\frac{\mid G(\omega) \mid^2}{T} 4A_1(1 - A_1) \cdot \left[\sin^2\left(\frac{\omega x_1 - \omega x_2}{2}\right)\right]$

$$\omega_p = \frac{2\pi}{T}, \qquad m = 0, \pm 1, \pm 2, \cdots$$

TABLE V(a)—RANDOM PULSE DURATION MODULATION, (PDM)—(RECTANGULAR PULSE)

Distribution Function	Discrete Spectral Power Density Function $P_D(\omega)$	Continuous Spectral Power Density Function, $P_C(\omega)$
Sinusoidal	$\frac{2\pi}{T^2} E^2\tau^2 \frac{\sin^2\left(\frac{\omega\tau}{2}\right)}{\left(\frac{\omega\tau}{2}\right)^2} J_0^2\left(\frac{\omega x_0}{2}\right)\delta(\omega - m\omega_p)$	$\frac{2E^2}{T\omega^2}\left[1 - \cos(\omega\tau) J_0(\omega x_0) - 2\sin^2\left(\frac{\omega\tau}{2}\right) J_0^2\left(\frac{\omega x_0}{2}\right)\right]$
Normal	$\frac{2\pi}{T^2} E^2\tau^2 \frac{\sin^2\left(\frac{\omega\tau}{2}\right)}{\left(\frac{\omega\tau}{2}\right)^2} \exp\left(\frac{-\sigma^2\omega^2}{4}\right)\delta(\omega - m\omega_p)$	$\frac{2E^2}{T\omega^2}\left[1 - \cos(\omega\tau)\exp\left(\frac{-\sigma^2\omega^2}{2}\right) - 2\sin^2\left(\frac{\omega\tau}{2}\right)\exp\left(\frac{-\sigma^2\omega^2}{4}\right)\right]$
Flat	$\frac{2\pi}{T^2} E^2\tau^2 \frac{\sin^2\left(\frac{\omega\tau}{2}\right)}{\left(\frac{\omega\tau}{2}\right)^2} \frac{\sin^2\left(\frac{\omega x_0}{2}\right)}{\left(\frac{\omega x_0}{2}\right)^2} \delta(\omega - m\omega_p)$	$\frac{2E^2}{T\omega^2}\left[1 - \cos(\omega\tau)\frac{\sin(\omega x_0)}{(\omega x_0)} - 2\sin^2\left(\frac{\omega\tau}{2}\right)\frac{\sin^2\left(\frac{\omega x_0}{2}\right)}{\left(\frac{\omega x_0}{2}\right)^2}\right]$
Double Spike	$\frac{8\pi E^2}{T^2\omega^2}\left[A_1\sin\left(\frac{\omega x_1}{2}\right) + (1 - A_1)\sin\left(\frac{\omega x_2}{2}\right)\right]^2 \cdot \delta(\omega - m\omega_p)$	$\frac{4E^2}{T\omega^2} A_1(1 - A_1) \left[\sin\left(\frac{\omega x_1}{2}\right) - \sin\left(\frac{\omega x_2}{2}\right)\right]^2$

$$\omega_p = \frac{2\pi}{T}, \qquad m = 0, \pm 1, \pm 2, \cdots$$

TABLE V(b)—RANDOM PULSE DURATION MODULATION, (PDM)—(Gaussian Pulse)

Distribution Function	Discrete Spectral Power Density Function, $P_D(\omega)$	Continuous Spectral Power Density Function, $P_C(\omega)$
Normal	$\frac{2\pi}{T^2}\frac{\pi E^2\tau^2}{4\ln 2}\left[\frac{8\ln 2}{\sigma^2\omega^2+8\ln 2}\right]^3 \cdot \exp\left(\frac{-\omega^2\tau^2}{\omega^2\sigma^2+8\ln 2}\right)\delta(\omega-m\omega_p)$	$\frac{E^2\pi\tau^2}{4T\ln 2}\left[\frac{\left\{1+\frac{\sigma^2}{\tau^2}\left(1+\frac{\sigma^2\omega^2}{4\ln 2}\right)\right\}}{\left(1+\frac{\sigma^2\omega^2}{4\ln 2}\right)^{5/2}}\exp\left(\frac{-\omega^2\tau^2}{8\ln 2+2\sigma^2\omega^2}\right) - \frac{1}{\left(1+\frac{\sigma^2\omega^2}{8\ln 2}\right)^3}\exp\left(\frac{-\omega^2\tau^2}{8\ln 2+\sigma^2\omega^2}\right)\right]$
Flat	$\frac{2\pi}{T^2}\left[\frac{E^2\pi 16\ln 2}{\omega^4x_0^2}\exp\left(\frac{-\omega^2\{\tau^2+x_0^2\}}{8\ln 2}\right) \cdot \sinh^2\left(\frac{\omega^2\tau x_0}{8\ln 2}\right)\right]\delta(\omega-m\omega_p)$	$\frac{E^2\pi}{2T\omega^2x_0}\left[2\exp\{-b^2(\tau^2+x_0^2)\}\{\tau\sinh(2b^2\tau x_0) - x_0\cosh(2b^2\tau x_0)\} + \frac{\sqrt{\pi}}{2b}\left\{\operatorname{erf}\left(\frac{\tau+x_0}{b}\right)-\operatorname{erf}\left(\frac{\tau-x_0}{b}\right)\right\} - \frac{4}{b^2x_0}\exp\{-b^2(\tau^2+x_0^2)\}\sinh^2(b^2\tau x_0)\right]$ where $b^2=\frac{\omega^2}{8\ln 2}$
Double Spike	$\frac{2\pi}{T^2}\left[\frac{\pi E^2}{4\ln 2}\right]\left[A_1x_1\exp\left(\frac{-\omega^2x_1^2}{16\ln 2}\right) + (1-A_1)x_2\exp\left(\frac{-\omega^2x_2^2}{16\ln 2}\right)\right]^2\delta(\omega-m\omega_p)$	$\frac{E^2}{T}\left(\frac{\pi}{4\ln 2}\right)A_1(1-A_1)\left[x_1\exp\left(\frac{-\omega^2x_1^2}{16\ln 2}\right) - x_2\exp\left(\frac{-\omega^2x_2^2}{16\ln 2}\right)\right]^2$

$$\omega_p=\frac{2\pi}{T},\qquad m=0,\pm 1,\pm 2,\cdots$$

TABLE V(c)—RANDOM PULSE DURATION MODULATION, (PDM)—(ERROR FUNCTION PULSE)

Distribution Function	Discrete Spectral Power Density Function, $P_D(\omega)$	Continuous Spectral Power Density Function, $P_C(\omega)$
Normal	$\frac{2\pi}{T^2}\frac{A^2\sin^2(aQ)}{2\sigma^2P}\exp\left(-\frac{2b^2\tau^2+a^2\sigma^2}{1+2\sigma^2b^2}\right)\delta(\omega-m\omega_p)$	$\frac{A^2}{T}\left\{\frac{N\left[1-\cos(2aR)\exp\left(\frac{-a^2}{B}\right)\right]}{2(1+4b^2\sigma^2)^{1/2}} - \frac{\sin^2(aQ)\exp\left(-\frac{2b^2\tau^2+a^2\sigma^2}{1+2\sigma^2b^2}\right)}{1+2\sigma^2b^2}\right\}$
Flat	$\frac{2\pi}{T^2}\frac{A^2\pi\exp\left(\frac{-a^2}{2b^2}\right)}{16b^2x_0^2}\left\{\operatorname{Im}\operatorname{erf}\left(b\tau-bx_0+\frac{ia}{2b}\right) - \operatorname{Im}\operatorname{erf}\left(b\tau+bx_0+\frac{ia}{2b}\right)\right\}^2\delta(\omega-m\omega_p)$	$\frac{A^2}{4\sqrt{2}Tbx_0}\left\{\operatorname{erf}[b\sqrt{2}(\tau+x_0)]-\operatorname{erf}[b\sqrt{2}(\tau-x_0)] + \frac{\pi^{1/2}}{2}\exp\left(\frac{-a^2}{2b^2}\right)\left[\operatorname{Re}\operatorname{erf}\left[b\sqrt{2}(\tau-x_0)+\frac{i\gamma}{2}\right] - \operatorname{Re}\operatorname{erf}\left[b\sqrt{2}(\tau+x_0)+\frac{i\gamma}{2}\right]\right]\right\} - \frac{A^2\pi\exp\left(\frac{-a^2}{2b^2}\right)}{16Tb^2x_0^2}\left\{\operatorname{Im}\operatorname{erf}\left(b\tau-bx_0+\frac{ia}{2b}\right) - \operatorname{Im}\operatorname{erf}\left(b\tau+bx_0+\frac{ia}{2b}\right)\right\}^2$

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TABLE V(c)—Random Pulse Duration Modulation, (PDM)—(Error Function Pulse)—*Concluded*

Distribution Function	Discrete Spectral Power Density Function, $P_D(\omega)$	Continuous Spectral Power Density Function, $P_C(\omega)$
Double Spike	$\frac{2\pi A^2}{T^2}\{A_1 \sin(ax_1)\exp(-b^2x_1^2) + (1 - A_1)\sin(ax_2)\exp(-b^2x_2^2)\}^2\delta(\omega - m\omega_p)$	$\frac{A^2A_1(1 - A_1)}{2T}\{\sin(ax_1)\exp(-b^2x_1^2) - \sin(ax_2)\exp(-b^2x_2^2)\}^2$

$$\omega_p = \frac{2\pi}{T}, \quad m = 0, \pm 1, \pm 2, \cdots \qquad A = \frac{2E}{\omega}, \quad a = \frac{\omega}{2}, \quad b^2 = \frac{\alpha^2\omega^2}{4\pi}, \quad \gamma = \frac{a\sqrt{2}}{b}$$

$$Q = \frac{\tau}{1 + 2\sigma^2b^2}, \quad N = \exp\left(\frac{-2b^2\tau^2}{1 + 4b^2\sigma^2}\right), \quad P = \frac{1 + 2\sigma^2b^2}{2\sigma^2}, \quad B = \frac{1 + 4\sigma^2b^2}{2\sigma^2}, \quad R = \frac{\tau}{1 + 4\sigma^2b^2}$$

TABLE VI—Random Single Edge Modulation, (SEM)—(Rectangular Pulse)

Distribution Function	Discrete Spectral Power Density Function, $P_D(\omega)$	Continuous Spectral Power Density Function, $P_C(\omega)$
Sinusoidal	$\frac{2\pi E^2}{T^2\omega^2}[1 + J_0^2(\omega x_0) - 2\cos(\omega\tau)J_0(\omega x_0)]\delta(\omega - m\omega_p)$	$\frac{E^2}{T\omega^2}[1 - J_0^2(\omega x_0)]$
Normal	$\frac{2\pi E^2}{T^2\omega^2}\left[1 + \exp(-\sigma^2\omega^2) - 2\cos(\omega\tau)\exp\left(\frac{-\sigma^2\omega^2}{2}\right)\right]\delta(\omega - m\omega_p)$	$\frac{E^2}{T\omega^2}[1 - \exp(-\sigma^2\omega^2)]$
Flat	$\frac{2\pi E^2}{T^2\omega^2}\left[1 + \frac{\sin^2(\omega x_0)}{(\omega x_0)^2} - 2\cos(\omega\tau) \cdot \frac{\sin(\omega x_0)}{(\omega x_0)}\right]\delta(\omega - m\omega_p)$	$\frac{E^2}{T\omega^2}\left[1 - \frac{\sin^2(\omega x_0)}{(\omega x_0)^2}\right]$
Double Spike	$\frac{4\pi E^2}{T^2\omega^2}\left[1 - A_1 + A_1^2 - A_1\cos\left(\omega\left\{x_1 + \frac{\tau}{2}\right\}\right) - (1 - A_1)\cos\left(\omega\left\{x_2 + \frac{\tau}{2}\right\}\right) + A_1(1 - A_1)\cos(\omega\{x_1 - x_2\})\right]\delta(\omega - m\omega_p)$	$\frac{2E^2}{T\omega^2}A_1(1 - A_1)[1 - \cos(\omega\{x_1 - x_2\})]$

$$\omega_p = \frac{2\pi}{T}, \qquad m = 0, \pm 1, \pm 2, \cdots$$

A Note on the Sampling Theorem

L. J. FOGEL†

Summary—**The human operator often perceives rate as well as amplitude information in sampling various displayed continuous parameters. It is therefore necessary to extend the Sampling Theorem to allow the analysis of certain man-machine relations. The result is stated and the required mathematics included in the Appendix. Certain distinct problem areas where this extension can be employed fruitfully are indicated.**

THE AIRCRAFT instrument display-to-pilot communication channel normally operates with the sequential scanning of the various individual indicators, each concerned with a different variable. It is because of this *modus operandi* that it is necessary to apply the Sampling Theorem when taking a theoretical approach toward the analysis and design of aircraft instrument displays. To relate the required human operation to measured human capabilities, it is essential to include the fact that the human operator often observes derivative as well as function amplitude value in each sampling. For example, in the case of pointer-on-scale displays, he may estimate the pointer position (which corresponds to function amplitude), the rate and possibly even some information concerning the acceleration of the pointer (corresponding to first- and second-time derivatives). Thus, it becomes necessary to extend the Sampling Theorem to determine the periodic sampling interval required to fully represent the transmitted frequency limited message when the instantaneous sampling includes the function amplitude as well as derivative values.

From the Appendix, it may be seen that when the first derivative, $f'(t_n)$, $(n = \cdots -2, -1, 0, +1, +2, \cdots)$ alone is added to the function amplitude sample information, $f(t_n)$, a surprisingly simple result is obtained, $T = 1/W$; that is, twice the interval required when $f(t)$ alone is observed. The addition of each succeeding derivative allows the time interval between samples to become larger, according to $T = (k+1)/2W$, where k is the order of the highest derivative when all lower-ordered derivatives are observed in each sample.

Further, it may be seen that T is a discrete increasing function of the *number*, M, of observed derivates. It should here be pointed out that the word "number" is correct, and is a true generalization of the $k+1$ numerator when all k lower-ordered derivatives are observed. Thus, if the observer determines $f(t_n)$ and $f'''(t_n)$ only, the proper T is still $1/W$. In general, the Extended Sampling Theorem may be stated as follows:

If a function $f(t)$ contains no frequency higher than W cycles per second, it is determined by giving M function derivate values at each of a series of points extending throughout the time domain, sampling interval $T = M/2W$ being the time interval between instantaneous observations.

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It is important to note that this result is not in conflict with the general statement that $2WT$ independent sample values are required to specify a function of duration T and bandwidth W. It merely indicates various ways in which these independent samples may be observed.

With this extended theorem established, various aspects of the aircraft instrumentation study program become accessible, and may be analyzed. From knowledge of the engineering aspects of flight and empirical data, it is possible to estimate the highest frequency component in the displayed parameter functions. The next step is to combine this data with the results of psychometric studies which approximately determine the human operator sampling process characteristics. An experimental study of human smoothing should reveal an estimate of the resultant observed signal-to-noise ratio, S/N, as a function of the received S/N during each sample. Since the observed S/N of higher ordered derivatives is lower due to judgment error "noise," there should be some optimal compromise in the number of derivatives observed which will result in the highest S/N for the sampled and smoothed resultant received message.

It is expected that further use will be made of this tool in human engineering analysis with regard to high performance aircraft flight control, ground vehicle control, and human operator target tracking. Careful consideration should also yield possible useful applications of the Extended Sampling Theorem in the fields of air traffic control simulation, data link computation, and telemetry.

Appendix

Consider $f(t)$ to be a function which admits of Fourier representation,[1] then

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} s(\omega)e^{i\omega t}\,d\omega. \tag{1}$$

If the frequency spectrum, $s(\omega)$, of this function is bounded such that $-W < \omega < +W$, then

$$f(t) = \frac{1}{2\pi}\int_{-2\pi W}^{+2\pi W} s(\omega)e^{i\omega t}\,d\omega. \tag{2}$$

The Sampling Theorem is derived from the fact that periodic sampling of this function, $f(t)$, at the points nT, where $T = 1/2W$, produces the sequence of Fourier co-efficients $f(n/2W)$, where $n = \cdots -2, -1, 0, +1, +2, \cdots$. Since $s(\omega)$ is nonzero in a finite interval, this sequence, $f(n/2W)$, determines $s(\omega)$ and thus $f(t)$.

[1] The function $f(t)$ may have only a finite number of points of discontinuity and a finite number of maxima and minima in any finite interval and

$$\int_{-\infty}^{+\infty} |f(t)|\,dt < \infty.$$

Suppose the sampling interval is $T = 1/W$; then the sequence of Fourier coefficients becomes

$$f\left(\frac{n}{W}\right) = \frac{1}{2\pi}\int_{-2\pi W}^{+2\pi W} s(\omega)e^{j(n/W)\omega}\,d\omega. \tag{3}$$

By a linear transformation of the sectionalized integral (folding), the following may be accomplished:

$$= \frac{1}{2\pi}\left\{\int_0^{\pi W} s(\omega)e^{j(n/W)\omega}\,d\omega + \int_{\pi W}^{2\pi W} s(\omega')e^{j(n/W)\omega'}\,d\omega' + \int_{-\pi W}^{0} s(\omega)e^{j(n/W)\omega}\,d\omega + \int_{-2\pi W}^{-\pi W} s(\omega')e^{j(n/W)\omega'}\,d\omega'\right\}. \tag{4}$$

Transform the second and fourth integrals according to $\omega' = 2\pi W + \omega$ and $\omega' = \omega - 2\pi W$ respectively then rearrange to give

$$= \frac{1}{2\pi}\left\{\int_0^{\pi W} [s(\omega) + s(\omega - 2\pi W)]e^{j(n/W)\omega}\,d\omega + \int_{-\pi W}^{0} [s(\omega) + s(\omega + 2\pi W)]e^{j(n/W)\omega}\,d\omega\right\}. \tag{5}$$

Since $s(\omega)$ is symmetric with respect to the origin, so also are both bracketed quantities in the integrand.

Suppose that the function derivative is also given, then, assuming it to be possible to differentiate under the integral sign,[2]

$$f'(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega s(\omega)e^{j\omega t}\,d\omega. \tag{6}$$

If the sampling is performed in a similar manner, at $t = n/W$, and the frequency spectrum remains limited to the same range $-W < \omega < +W$, then as in (5)

$$f'\left(\frac{n}{W}\right) = \frac{j}{2\pi}\left\{\int_0^{\pi W} [\omega s(\omega) + (\omega - 2\pi W)s(\omega - 2\pi W)]e^{j(n/W)\omega}\,d\omega + \int_{-\pi W}^{0} [\omega s(\omega) + (\omega + 2\pi W)s(\omega + 2\pi W)]e^{j(n/W)\omega}\,d\omega\right\}. \tag{7}$$

The sequence $f(n/W)$ and $f'(n/W)$ give the simultaneous equations

$$\text{and}\quad \left.\begin{aligned} \alpha_0(\omega) &= s(\omega) + s(\omega - 2\pi W) \\ \alpha_1(\omega) &= \omega s(\omega) + (\omega - 2\pi W)s(\omega - 2\pi W) \end{aligned}\right\} \tag{8}$$

in the positive region of ω, where α_k $(k = 0, 1)$ is the spectrum determined by the discrete set of points if the sampling length is taken as fundamental frequency, i.e.

$$f^k\left(\frac{n}{W}\right) = \frac{j^k}{2\pi}\int_{-\pi W}^{\pi W} \alpha_k(\omega)e^{j(n/W)\omega}\,d\omega. \tag{9}$$

$s(\omega)$ can be determined from (8), thus $f(t)$ can be determined.

[2] A sufficient condition for this is that this integral,

$$\int_{-\infty}^{\infty} j\omega s(\omega)e^{j\omega t}\,d\omega$$

converges uniformly and that the original integral in equation also converges (which has already been assumed).

In a similar manner, if $T = M/2W$ where $M = k + 1$ and k is the highest order of the complete derivative set which is taken, then

$$f(nT) = f\left(\frac{nM}{2W}\right) = \frac{1}{2\pi}\sum_{r=0}^{M-1}\left\{\int_{2\pi Wr/M}^{[2\pi W(r+1)]/M} s(\omega)e^{j(nM/2W)\omega}\,d\omega + \int_{-[2\pi W(r+1)]/M}^{2\pi Wr/M} s(\omega)e^{j(nM/2W)\omega}\,d\omega\right\} \tag{10}$$

where $r = 0, 1, 2, \cdots (M - 1)$ and

$$f^k\left(\frac{nM}{2W}\right) = \frac{j^k}{2\pi}\sum_{r=0}^{M-1}\left\{\int_0^{2\pi W/M} (-1)^{nr}\left(\omega - \frac{2\pi Wr}{M}\right)^k \cdot s\left(\omega - \frac{2\pi Wr}{M}\right)e^{j(nM/2W)\omega}\,d\omega + \int_{-2\pi W/M}^{0} (-1)^{nr}\left(\omega + \frac{2\pi Wr}{M}\right)^k s\left(\omega + \frac{2\pi Wr}{M}\right)e^{j(nM/2W)\omega}\,d\omega.\right. \tag{11}$$

Each sequence $f^k(nM/2W)$ determines a function $\alpha_k(\omega)$ defined by

$$f^k\left(\frac{nM}{2W}\right) = \frac{j^k}{2\pi}\int_{-2\pi W/M}^{2\pi W/M} \alpha_k(\omega)e^{j(nM/2W)\omega}\,d\omega, \tag{12}$$

such that $-2\pi W/M < \omega < +2\pi W/M$. A set of simultaneous equations may be set up from the general expression,

for $\omega > 0$

$$\alpha_k(\omega) = \sum_{r=0}^{M-1} (-1)^{nr}\left(\omega - \frac{2\pi Wr}{M}\right)^k s\left(\omega - \frac{2\pi Wr}{M}\right), \tag{13}$$

where k and $r = 0, 1, 2, \cdots . (M - 1)$, that is,

$$\left\{\begin{aligned} \alpha_0(\omega) &= s(\omega) + (-1)^n s\left(\omega - \frac{2\pi W}{M}\right) + s\left(\omega - \frac{4\pi W}{M}\right) \\ &\quad \cdots + (-1)^{nr} s\left(\omega - \frac{2\pi Wr}{M}\right) + \cdots \\ &\ \ \vdots \\ \alpha_{M-1}(\omega) &= \omega^{M-1}s(\omega) + (-1)^n\left(\omega - \frac{2\pi W}{M}\right)^{M-1} s\left(\omega - \frac{2\pi W}{M}\right) \\ &\quad + \left(\omega - \frac{4\pi W}{M}\right)^{M-1} s\left(\omega - \frac{4\pi W}{M}\right) + \cdots . \\ &\quad + (-1)^{nr}\left(\omega - \frac{2\pi Wr}{M}\right)^{M-1} s\left(\omega - \frac{2\pi Wr}{M}\right) + \cdots . \end{aligned}\right. \tag{14}$$

A unique solution exists for $s(\omega - 2\pi Wr/M)$ in terms of $\alpha_k(\omega)$ if the determinant of the coefficients is not zero, which then determines $f(t)$.

If the derivatives are not observed at the same time instants, it may easily be seen that this makes no change in the above derivation, since the Shifting Theorem allows the time shift to be expressed as a single exponential factor in the frequency domain

$$f(t + \epsilon) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{j\omega\epsilon}s(\omega)e^{j\omega t}\,d\omega. \tag{15}$$

The constructed set of $M - 1$ simultaneous equations will, in general, still be solvable under this change.

Statistical Calculation of Word Entropies for Four Western Languages

G. A. BARNARD, III†

Summary—**Using a modified version of Shannon's method,[1] comparative figures for the word-letter entropies of printed English, French, German, and Spanish are obtained and the method described.**

Review of Shannon's Method

Using his now universal definition of entropy,[2] Shannon has shown how the character entropy of a printed language can be calculated by deducting the asymptotic value approached by a series of approximations to the entropy. These approximations are made by solving for the entropy in each case, using successively a conditional probability involving one more letter of preceding text. That is to say,

$$F_N = -\sum_{i,j} p(b_i\,,j)\log_2 p_{b_i}(j)$$

$$= -\sum_{i,j} p(b_i\,,j)\log_2 p(b_i\,,j) + \sum_i p(b_i)\log_2 p(b_i);$$

where F_N is the entropy calculation per letter for the Nth letter (based on conditional probabilities through $(N-1)$ letters), b_i is a block of $(N-1)$ letters, j is the Nth letter, and $p_{b_i}(j)$ is the probability of the Nth letter, j, with conditional probability dependency on the contents of b_i.

Thus F_N is a measure of the average information content of the Nth letter in a text when the $(N-1)$ preceding letters, with their statistical probabilities, are all known and thus, theoretically, the entropy, $H = \lim_{N\to\infty} F_N$.

For any language using the Roman alphabet, consisting of twenty-six letters, not considering punctuation or spaces,

$$F_0 = \log_2 26 = 4.70.$$

For English, Shannon completed statistical calculations of entropy for $(b_i + j)$ equaling from one to three; i.e., F_1, F_2, F_3. These are the values for the computation of which frequency tables are available.[3] Tables are also available for English word frequencies;[4] Shannon used these, also, to obtain a fourth value, F_{word}, or F_W, of entropy. He presented arguments for showing $F_W \cong F_8$, although average word length, for English, is 4.5 letters.

Rather than attempt the colossal task of compiling tables for further values, Shannon used an English speaking subject as an experimental "predictor" to obtain values of F_4 through F_{15}, and F_{100}. Armed with probability tables for two and three letter combinations, plus his life's experience with English, its grammar and word order, this person made calculated guesses of the "next letter," j, in the different length phrases. His correct and incorrect guessing helped furnish data from which Shannon could show enough values of F_N to get a good asymptotic approximation for H.

The word list which Shannon used for the calculations to obtain F_W is a list[4] of 1,027 words composed from sampling 100,000 words of printed English text. This letter entropy per word was evaluated as

$$F_W = -\sum_{n=1}^{M} p_n \log_2 p_n \tag{1}$$

where

$$p_n \cong \frac{k}{n} \tag{2}$$

and

$$\sum_{n=1}^{M} p_n = 1, \tag{3}$$

for consistency, assuming that the list ends at n equal to a value such that the total of all probabilities is unity.

Shannon then plotted his word-rank versus word-frequency on log-log paper and by inspection found $k \cong 0.1$.

Examination of his results shows his method to have been the solving for M in (3) by the approximation

$$\sum_{n=1}^{M} p_n \cong \sum_{n=1}^{J} p_n + \int_{J.5}^{M} p_n\,dn = 1. \tag{4}$$

Then he appears to have solved for the letter entropy per word by

$$F_W = \frac{1}{\alpha}\left(-\sum_{n=1}^{M} p_n \log_2 p_n\right),$$

approximated as

$$F_W \cong -\frac{1}{\alpha}\left[\sum_{n=1}^{J} p_n \log_2 p_n + \int_{J.5}^{M} p_n \log_2 p_n\,dn\right]$$

$$= -\frac{1}{\alpha}\left[\sum_{1}^{J} \frac{k}{n}\log_2\frac{k}{n} + \int_{J.5}^{M} \frac{k}{n}\log_2\frac{k}{n}\,dn\right]$$

$$= -\frac{1}{\alpha}\left[\log_2 k \sum_{1}^{J}\frac{k}{n} - k\sum_{1}^{J}\frac{\log_2 n}{n}\right.$$

$$\left. + \log_2 k\int_{J.5}^{M}\frac{k}{n}\,dn - k\int_{J.5}^{M}\frac{\log_2 n}{n}\,dn\right]$$

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[1] C. Shannon, "Prediction and entropy of printed English," *Bell Sys. Tech. Jour.*, vol. 30, pp. 50–64; January, 1951.

[2] C. Shannon, "A mathematical theory of communication," *Bell Sys. Tech. Jour.*, vol. 27, pp. 379–423; July, 1948, pp. 623–656; October, 1948.

[3] F. Pratt, "Secret and Urgent," Blue Ribbon Books, New York, N. Y.; 1942.

[4] G. Dewey, "Relativ Frequency of English Speech Sounds," Harvard University Press, Cambridge, Mass.; 1923.

$$= -\frac{1}{\alpha}\left[\log_2 k\left(\sum_1^J \frac{k}{n} + \int_{J.5}^M \frac{k}{n}\,dn\right) - k\left(\sum_1^J \frac{\log_2 n}{n} + \int_{J.5}^M \frac{\log_2 n}{n}\,dn\right)\right]$$

and since, in (4),

$$\sum_1^J \frac{k}{n} + \int_{J.5}^M \frac{k}{n}\,dn = 1,$$

$$F_W = -\frac{1}{\alpha}\left[\log_2 k - k\left(\sum_1^J \frac{\log_2 n}{n} + \int_{J.5}^M \frac{\log_2 n}{n}\,dn\right)\right], \quad (5)$$

where $\alpha = 4.5$ is the average number of letters per word for English and $J = 1{,}027$ is the list length.

Except that it is an interesting figure to know, the direct solution for M could have been eliminated, since, in both (4) and (5), the integral portion of the solutions contains $\ln M$ as the only remaining unknown. Thus (5) would be

$$F_W = -\frac{1}{\alpha}\left[\log_2 k - k\left(\sum_1^J \frac{\log_2 n}{n} + 1.44269 \int_{J.5}^M \frac{\ln n}{n}\,dn\right)\right]$$

since $\log_2 n = 1.44269 \ln n$

$$\therefore F_W = -\frac{1.44269}{\alpha}\left[\ln k - k\left(\sum_1^J \frac{\ln n}{n} + \left\{\frac{(\ln M)^2}{2} - \frac{(\ln J.5)^2}{2}\right\}\right)\right], \quad (6)$$

and (4) would be

$$\sum_{n=1}^J \frac{k}{n} + k[\ln M - \ln J.5] = 1$$

or

$$\ln M = \frac{1 - \sum_1^J \frac{k}{n}}{k} + \ln J.5$$

$$= \frac{1}{k} - \sum_1^J \frac{1}{n} + \ln J.5. \quad (7)$$

Shannon's examination of his log-log plot gave him $k = 0.1$, the word list gave

$$\sum_{n=1}^{1027} \frac{k}{n} = .78633;$$

and the expression

$$\sum_{n=1}^{1027} \frac{\ln n}{n}$$

can be computed from the word list data.

What Shannon's method amounts to, then, is, in (1), taking the sum of the actual experimental values in a 100,000 word total count of $(p_n \log_2 p_n)$ out to a word-rank limited by the extent of the word list, then taking the sum of the mathematically computed values of this same expression out to a value of n, word-rank, where the total probability of all words equals unity; p_n being figured as equal to k/n, and k estimated from a graphical plot of the word list probabilities versus ranks.

Comparative Method of Computation with Rooted Word Lists

Should equivalent word-frequency lists be available for other languages, particularly ones using the Roman alphabet (or any other of 26 letters), a valuable comparison could be made of the entropies (and thus redundancies) of these languages. Knowledge of which languages had the highest entropies (or lowest redundancies) could be a guide towards shortening message lengths for given information content or towards synthesis of the most efficient machine language for translation devices or for computing machines.

The list used by Shannon for English was one making full distinction between all words of different spelling. For example, the words "I, I'd, I'll, I'm, would, will, am" are listed as seven discrete words, the listing for the word "would" not containing a count of its apostrophized counterpart in "I'd."

For this paper, investigation has been made of French, German, and Spanish, as these and English are the four principal languages of the western (Roman alphabetized) world today. However, word frequency lists of the same type as the English one above are not available in these three languages; rather, there are lists of words grouped according to roots,[5] for example, all conjugated forms of a verb being listed under the infinitive as one composite word listing. Such a list is also available for English. The original reason for compiling word lists of any kind was for pedagogical purposes, and, although the pure lists must have existed first in the compilation process, they were not published, since the rooted list is the one most useful for the teaching of languages.

Investigation of the problem of compiling the desired lists showed that the task for only one language would be out of the question for this paper. In compiling his list for English, Dewey says, "The word counting stage was expedited and checkt as far as possible by mechanical and other aids, including specially printed record cards, Veeder counters, Rand card holders, Smith steel signals, and a special card ledger desk. This stage, with fonetic transcription on the original cards which accompanied it required in all about 1200 hours."[4] This is the same as five months at eight hours per day, seven days per week. The same work could now be done on an IBM punched-card calculator for an estimated total cost of $500 (materials, operator labor, machine rental) and a minimum time of one month (eight hour days, five days a week), for a total count of 100,000 words.

It was decided to make a first step in the comparison process by using the rooted lists available, thereby getting a rough comparison of the entropies and pointing out a method of approach for better comparison in later work. The basic assumption necessary in accepting these lists as

[5] B. Q. Morgan, "German Frequency Word Book," The Macmillan Company, New York, N. Y.; 1928.
G. E. VanderBeke, "French Word Book," The Macmillan Company, New York, N. Y.; 1929.
M. E. Buchanan, "A Graded Spanish Word Book," The University of Toronto Press, Toronto, Canada; 1927.

the bases of comparison is that they have each covered the same general fields of literature with equal proportioning between the various literature fields. And equally important is that the combining into root groups is done in an equivalent manner with each list. This latter is undoubtedly the less certain for these lists.

The results, based on a pure 26-letter alphabet, are as follows:

	English	*French*	*German*	*Spanish*
F_0	4.70	4.70	4.70	4.70
F_1	4.124	3.984	4.095	4.015
F_W	1.648	3.02	1.08	1.97
	(w = 4.5)[3]	(w = 4.84)[3]	(w = 5.92)[3]	(w = 4.96)[3]

Tables[3] are available for the single letter frequencies of the four languages, so that the computations of F_1 come simply from

$$F_1 = -\sum_{i=1}^{26} p_i \log_2 p_i \ .$$

The method used for solving for F_W is as follows:

1. Plot a graph of word frequency (probability) versus word-rank on log-log paper.

2. From the graph, determine k as defined in (2). (This mathematical relationship between p_n and n is used, based on a statement by Zipf[6] that it is a good approximation for many languages.) The complete graph should not be used to estimate k since often it is much more nearly approaching its asymptote after a rank, n, of several hundred. In this paper the range from $n = 200$ to $n = 800$ was used, principally because the four lists to be compared contained specific data through this range. The range for estimation should not be too near the low-p_n end of the graph either, since the accuracy of the probability figure gets lower as n increases, for any given total word count. For example, the probability figure of 0.0731 for the English word "the" (7,310 occurrences out of 100,000) is much more reliable than the figure 0.00011 for the word "worse" (11 occurrences out of 100,000). That is to say, the smaller the number of occurrences within a given text sampling, the more the probability figure depends upon the character of the particular text samples chosen for the test. The word "worse" could easily have occurred only 5 or 6 times out of 100,000 if a slightly different group of texts had been chosen for the tests—thus getting its probability figure to be one-half what it was with 11 occurrences. This is illustrated by noting that at their far end two of the graphs plotted herein begin to drop down and k gets smaller.

3. Having obtained k, we now dispense with the lists and curves, some of which, incidentally, have no data for the first several word-ranks, and assume a pure hyperbolic curve beginning at $n = 1$ and ending at the value of n where $\sum_n p_n = 1$. Thus we use (6) and (7) derived above, but with mathematically determined values, rather than empirical ones, for the expression $\sum k/n$ and $\sum \ln n/n$; the upper limit of these sums can be as high as needed to stay within any desired degree of accuracy when the integrating approximation is used for the end portion of the evaluation. This limit can even be chosen as equal to M, no integration then being needed. For this paper, the limit was chosen arbitrarily as 100, giving us the expressions for (6) and (7) as

$$J = 100$$

$$\sum_{n=1}^{J} \frac{1}{n} = 5.1874$$

$$\ln J.5 = 4.61016$$

$$\ln M = \frac{1}{k} - .57724$$

$$F_W = \frac{1}{\alpha}\left(-1.44269 \ln k + .12661\, k + \frac{.721345}{k} - .83278\right).$$

The principal problem was the obtaining of the figures for p_n in each language. For English, this was easy since Dewey's rooted list is tabulated by order of greatest occurrence and the number of occurrences out of 100,000 are given for each word, so that $p_n = f_n/100{,}000 = f_n \times 10^{-5}$. These values were plotted directly for the points n, shown in Fig. 1.

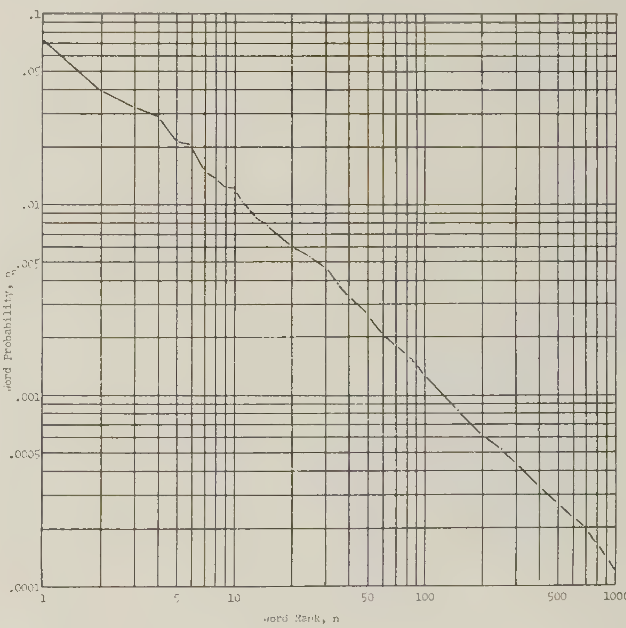

Fig. 1—Probability—rank curve for English words (rooted data).

$$k\Big]_{n=200}^{800} = .135.$$

For German, Morgan's adaptation of Kaeding's list groups the rooted words into blocks lying between two frequency limits, giving us accurate frequency values for only first and last words of the block. These were taken from Kaeding's total count of 10,910,777 (See Fig. 2).

For French, VanderBeke gives figures for Henmon's original list, based on a count of 400,000 through $n = 63$. For $n > 63$, VanderBeke's count is 1,147,748. The list is

[6] G. K. Zipf, "Human Behavior and the Principle of Least Effort," Addison-Wesley Press; 1949.

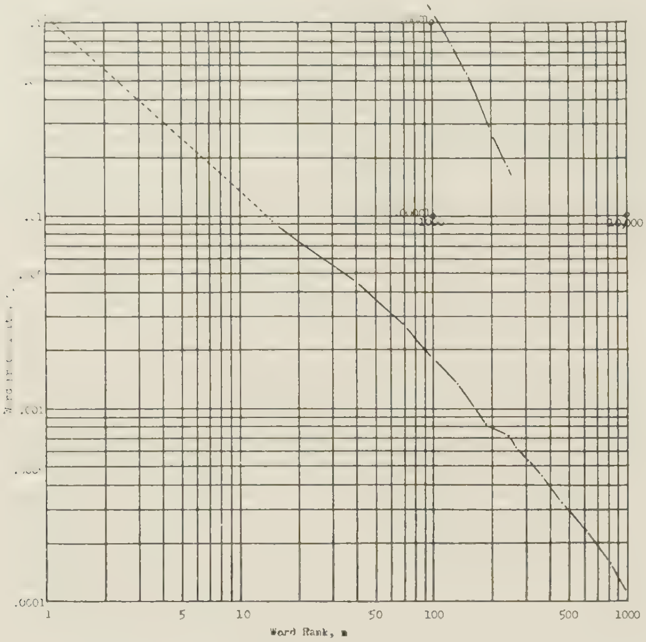

Fig. 2—Probability—rank curve for German words (rooted data). $k\big]_{n=200}^{800} = .157$.

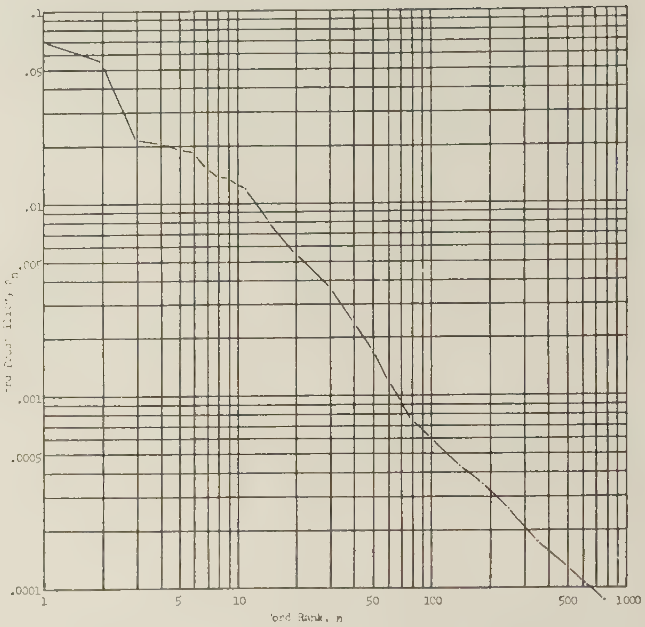

Fig. 3—Probability—rank curve for French words (rooted data). $k\big]_{n=200}^{800} = .063$.

published alphabetically, so that the figures were obtained by sorting according to frequency as in Fig. 3.

For Spanish, Buchanan published his frequency list in a modified form whereby he gives a weighted value to the frequency figure, the weighting depending upon how many of the different separate sample texts contained the word; his figure was $(f/10 + r)$ where f is the actual frequency of occurrence out of the count of 1,200,000 words and r is "range," or the number of text samples containing the word. He also published an alphabetical list of his 6,513 different words, giving both f and r. Rather than rearrange this whole list to obtain the few points needed for the graph, in this paper the method used for determining p_n was that of picking a word, n, in the weighted-frequency list and, from the alphabetical list, determining its f together with the f's of the five words preceding it in the frequency list and the five words following it; that is,

$$p_n = \frac{\frac{1}{11}\sum_{i=n-5}^{n+5} f_i}{1{,}200{,}000}.$$

Buchanan made his list with the effort-saving assumption that words of ranks $n = 1$ through $n = 189$ were important enough so that, for pedagogical purposes, they need not be counted. Thus Fig. 4 (next page) gives no data at all for the first part of the curve.

Comparison of the Two Methods

Had Shannon used the method herein described for solving for F_W, using, as he did, the nonrooted list for English, he would have used (8) with $\alpha = 4.5$, $k = .103$ as obtained from Fig. 5 (next page). This graph is plotted from the nonrooted word-frequency list. Then F_W would have been:

$$F_W = \frac{1}{4.5}\left(-1.44269 \ln .103 + .12661 \times .103 + \frac{.721345}{.103} - .83278\right) = 2.103.$$

Shannon's direct method obtained $F_W = 2.62$, the difference being due chiefly to the use in this paper, in (6) and (7), of mathematically derived data out to only $J = 100$ whereas Shannon used *statistical* data out to $J = 1{,}027$, and to a certain extent to the difference between the values of k. The smaller the k value, the larger the entropy.

Comparison of the Use of Rooted vs Unrooted Data

It is well to note at this point that F_W is greater for unrooted data than for rooted data. As we see above,

$$\text{For English} \begin{cases} F_{W\text{rooted}} = 1.648 & \text{where } k = .135 \\ F_{W\text{nonrooted}} = 2.103 & \text{where } k = .103. \end{cases}$$

This is because, in (8), k being less than unity, the increase of the first and third terms with decreasing k is much greater than the decrease in the second term; hence the increase of F_W.

It seems reasonable to deduce the assumption that the same situation would be true in the other three languages. Certainly the average values of p_n would increase for given values of n since the occurrence rate of a rooted word equals the sum of the occurrences of all words for which it is the composite. That is to say, since p_n increases, k must increase for given n; $p_n \cong k/n$ [see (2)].

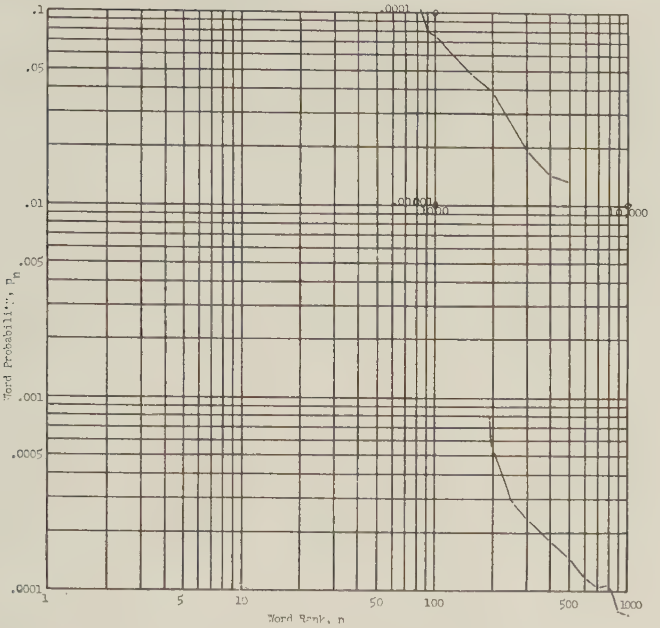

Fig. 4—Probability—rank curve for Spanish words (rooted data). $k]_{n=200}^{800} = .074.$

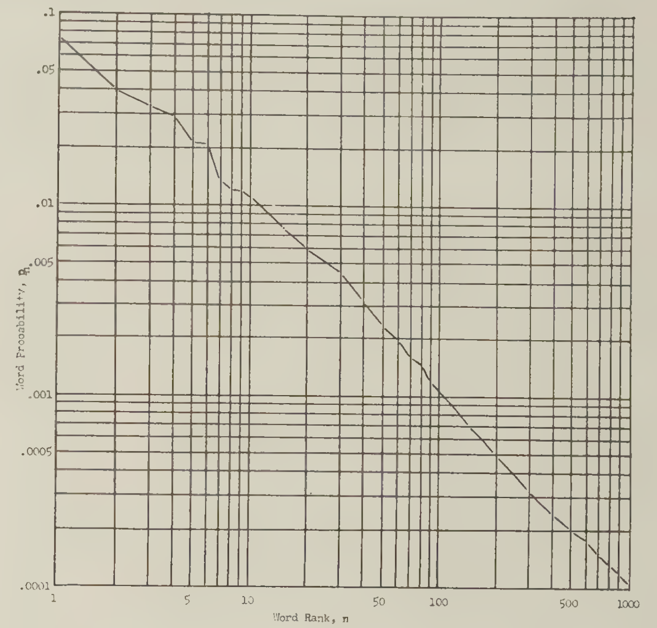

Fig. 5—Probability—rank curve for English words (nonrooted data). $k]_{n=200}^{800} = .103.$

Discussion of Results

The first point to note, in the results, is the fact that the average word length bears directly upon the values of F_W. It would be expected, then, to find F_W smaller for German than for the others since the German words are inordinately longer than in the other languages considered. The trend of smaller differences in word length between the other three languages themselves, however, does not follow this discussion for German.

To account, then, for the other differences in F_W between the languages, our immediate conclusion is that French must have a smaller vocabulary for expressing the same ideas as the others, i.e.: it appears to have a few explicit words, in contrast to many different words in other languages, to express the same idea. This reasoning could apply to German, also, in addition to the actual word-length as discussed above; the deduction being that German has less explicit words and needs to combine several to express an idea. We know that this is actually the reason for the long word-length in German: its long words are almost always found to be several small words combined into one. A check on this idea, which is beyond the scope of this paper, would be to compare the total number of words needed by each of the four languages for translation from an identical passage written in a fifth language; for example, the New Testament of the Bible translated from Greek or Aramaic.

For one lacking the conviction that French is the most explicit—or so much more so—of the four languages, the obvious step to take is to examine the validity of the two basic assumptions made above before accepting our data lists. Since we have found how much the value of F_W depends on k, and how much k depends upon the way in which the rooted words are grouped together, (in effect the larger the grouping together, the more explicit we are making the vocabulary) the more we realize how important it is for the root grouping to be equivalent in our data of the languages compared. Equivalence of literature fields covered is fairly easy to check, by examining the lists of the works from which the sample texts were taken.

Shannon's paper[1] brings out the upper and lower bounds obtainable experimentally (as described above) for the various values of F_N. His result for F_W in English, using nonrooted data, falls on the upper bound. The result for English found in this paper, using rooted data, falls near the lower bound.

Acknowledgment

The author wishes to express his appreciation to W. W. Harman and W. H. Kautz for their helpful suggestions on the organization of the material for this paper.

On the Modulation Levels in a Frequency Multiplexed Communication System by Statistical Methods*

R. L. BROCK† AND R. C. McCARTY†

Summary—This paper presents a mathematical analysis with experimental verification of the distribution of the instantaneous voltage of a complex signal resulting from the combination at random of a small number n of sinusoidal oscillations. The resulting calculated distributions are plotted in the form of a set of probability curves for comparison with curves obtained by experiment. Further laboratory measurements in which the individual sinusoidal oscillators are frequency-modulated in a manner suitable for communicating information in a binary form, yield substantially no change in the amplitude distribution as determined for the unmodulated oscillators. Consequently, the results of the mathematical analysis may be applied in the determination of M, the degree to which each subcarrier may amplitude modulate a final carrier in a FM-AM frequency multiplexed system. M may be determined for any desired degree of overmodulation in excess of one per cent and for as many subcarriers as are required in the system. Modulation levels determined according to approximate methods and the method described here are tabulated and compared.

Introduction

Multiplexed transmission of information using double and triple modulation systems has assumed increasing importance in recent years. Such systems may employ either time or frequency division. In frequency division, which is of interest here, individual subcarriers spaced in frequency may be amplitude-modulated (AM), frequency-modulated (FM), or single sideband-modulated (SS); groups of subcarriers may be combined to amplitude modulate or phase modulate a carrier.

One of the more interesting problems arising in the design of a frequency multiplex transmission system in which the final carrier is amplitude-modulated is that of determining the allowable degree to which the individual subcarriers may amplitude modulate the final carrier. Since frequency modulation results in a varying frequency sine wave of constant amplitude, it will be convenient mathematically to consider an FM-AM frequency multiplex system. That is, a system in which a group of equal amplitude, frequency-modulated subcarriers are linearly combined to form a complex signal that is used to amplitude modulate a higher frequency carrier. In order to insure a high signal-to-noise ratio for a particular subcarrier signal, it is necessary to make M, the factor by which each subcarrier modulates the main carrier, as large as possible. The optimum choice for M will depend on the system distortion that can be tolerated due to overmodulation and the distribution of the instantaneous voltage of the complex signal resulting from the combination of all subcarriers. It is the determination of the latter that is of primary interest here.

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Determination of Modulation Levels

In a recent paper Landon[1] indicates that for an FM-AM system the modulation index M should be determined according to

$$M = n^{-1} \text{ for small } n, \tag{1}$$

$$M = (1/8n)^{1/2} \text{ for large } n, \tag{2}$$

where n is the number of subcarriers and M is as previously defined. Choosing M according to these relations, effectively assures that overmodulation will not occur.

Eq. (1) is based on a linear summation of the individual subcarrier amplitudes. Eq. (2) is a result of the following reasoning: In another paper Landon[2] has shown that the distribution of voltage for fluctuation noise is normal and has a crest factor of about 4. That is, the ratio of the amplitude of the highest peaks to the rms value is 4. Peaks higher than this do occur but are highly improbable. If it is assumed that for large n the sum of n frequency-modulated equal amplitude subcarriers behaves like fluctuation noise, then the probability that the absolute value of the instantaneous voltage of the resultant signal exceeds γ times the rms amplitude (standard deviation) is

$$P(\gamma) = 2(2\pi)^{-1/2} \int_{\gamma}^{\infty} e^{-x^2/2}\, dx. \tag{3}$$

The rms amplitude is $(n/2)^{1/2}$ times the peak amplitude E of a single subcarrier, and if M is chosen to give 100 per cent modulation when the instantaneous voltage is γ times the rms value, then

$$\gamma(n/2)^{1/2}E = E_c\ , \tag{4}$$

where E_c is the peak amplitude of the unmodulated carrier. Since M is E/E_c it follows that

$$M = \gamma^{-1}(2/n)^{1/2}. \tag{5}$$

If γ is assigned the value 4, then (5) becomes (2), and from (3), which may be evaluated from tables of probability functions, the probability that the instantaneous voltage of the complex signal will exceed 4 times the rms value is essentially zero ($P = 0.000064$).

It is possible to choose $\gamma < 4$ in which case (5) permits a larger modulation level and (3) gives the corresponding probability of overmodulating or, on the average, the percentage of time overmodulation occurs. The resulting overmodulation appears to the receiver as noise and the degree to which this is tolerable depends on the noise

[1] V. D. Landon, "Theoretical analysis of various systems of multiplex transmission," *RCA Rev.*, vol. 9, pp. 287–351; June, 1948.
[2] V. D. Landon, "The distribution of amplitude with time in fluctation noise," Proc. I.R.E., vol. 29, pp. 50–55; February, 1941.

power present from other sources and the resultant degree of loss of information.

As the modulation level M is increased, the signal-to-noise power ratio S/N for a given subcarrier will increase. Even for M sufficiently large to cause overmodulation on the peaks, the S/N should continue to increase with M until the resulting overmodulation noise becomes comparable to the thermal noise of the receiver in its environment. Beyond this the S/N may be expected to decrease with increasing M. It is apparent that allowing some overmodulation to occur can offer real gains in the single subcarrier S/N.

Landon[3] has suggested that the normal distribution which should be employed above is $n = 8$ to 12, but for smaller n the linear summation should be used. The use of linear summation (while perhaps suitable in some applications) is not based on knowledge regarding the distribution function for small n. The value of n below which the normal distribution no longer describes the situation accurately can only be determined by a comparison with the true distributions. In what follows, the true distributions for small n will be approximated mathematically and verified experimentally.

Mathematical Analysis

For the case of frequency modulation, where the sinusoids are of constant amplitude, the resultant instantaneous value of multiplexing n subcarrier signals simultaneously from n independent sources may be expressed as

$$E_n = E \sum_{k=1}^{n} \cos(\omega_k t + \phi_k) \tag{6}$$

or

$$E_n = E \sum_{k=1}^{n} \cos \theta_k, \tag{7}$$

where $\theta_k = (\omega_k t + \phi_k)$. All values of the initial phase angle ϕ_k are equally probable in the interval $-\pi \leq \phi_k \leq \pi$.

The behavior of the instantaneous values E_n for a small number of subcarriers, (i.e.) $n \leq 12$, which is of particular interest here, can be readily determined by the methods of statistical analysis.

To determine the statistical properties of E_n, it is necessary to examine the distribution of $E \cos \theta$. θ in (7) assumes all values in the interval $-\pi \leq \theta \leq \pi$, with equal probability. Thus it is seen that the absolute value of $E \cos \theta$ is distributed in the interval, $0 \leq |E \cos \theta| \leq E$. If the individual sinusoids are combined at random, then E_n assumes a value at any time t, represented by the sum of n items drawn at random from a parent population whose distribution is that of $E \cos \theta$. When θ assumes values in the interval bounded by 0 and π, $E \cos \theta$ takes all values in the interval whose limits are $\pm E$, thus it is unnecessary to consider the entire range of variation of θ.

The probability of any value of θ in its range $0 \leq \theta \leq \pi$, is $d\theta/\pi$. Thus, for the parent population, we consider the general expression $y = E \cos \theta$, which yields,

$$\frac{d\theta}{\pi} = -\frac{dy}{\pi(E \sin \theta)}. \tag{8}$$

The density function $f(\theta)$ for $E \cos \theta$ is then

$$f(\theta) = -\frac{1}{\pi(E \sin \theta)} \tag{9}$$

and therefore

$$f(y) = -\frac{1}{\pi(E^2 - y^2)^{1/2}}, \tag{10}$$

the minus sign being neglected as all $\pm$ values of y are equally probable.

The representation of (10) by one of the types of density functions in the system developed by Karl Pearson,[4] provides a unique method of treating the statistical properties of the instantaneous values of y, and consequently E_n, when applied in conjunction with the works of Tchebycheff and Church.

In the Pearson system, only the first four moments are required to determine the nature of any distribution, with the exception of some degenerate types which may be determined by a smaller number.

Kendall[5] has shown that a distribution is uniquely determined by its moments if $\overline{\lim}_{n\to\infty} \nu_n^{1/n}/n$ is finite. As a corollary, a distribution is unique when its range is finite. For example, given the fact that the range of the distribution is from a to b and taking the origin at $x = a$, with the interval $b - a = c$,

$$\mu_n' = \int_a^b x^n f(x)\, dx \leq c^n, \tag{11}$$

therefore

$$(\mu_n')^{1/n} = (\nu_n)^{1/n} \leq c, \tag{12}$$

and thus $\overline{\lim}_{n\to\infty} \nu_n^{1/n}/n = 0$ where ν_n is the absolute nth moment. The range of the distribution whose density function is $f(y)$ is obviously finite and therefore the conditions for uniqueness are fulfilled. The first four moments of $f(y)$ are determined by the evaluation of

$$\mu_n' = \int_{-E}^{+E} (y)^n f(y)\, dy \tag{13}$$

and

$$\mu_n = \int_{-E}^{+E} (y - \mu_1')^n f(y)\, dy, \tag{14}$$

which yield

$$\mu_1' = 0, \qquad \mu_2 = \frac{E^2}{2}, \qquad \mu_3 = 0, \qquad \mu_4 = \frac{3E^4}{8}. \tag{15}$$

The second, third and fourth moments were taken about the mean, μ_1'. The moments of odd order vanish because

[3] V. D. Landon, "Theoretical analysis of various systems of multiplex transmission," *RCA Review*, vol. 9, pp. 433–482; September, 1948.

[4] M. G. Kendall, "The Advanced Theory of Statistics," Hafner Publishing Co., New York, N. Y., vol. 1, pp. 137–143; 1952.

[5] *Ibid.*, pp. 108–109.

the distribution is symmetrical about the mean. The parameters of the distribution β_1 and β_2 are defined in terms of the three moments μ_2 to μ_4 by

$$\beta_1 = \mu_3^2/\mu_2^3, \qquad \beta_2 = \mu_4/\mu_2^2. \tag{16}$$

Evaluation of (16) gives $\beta_1 = 0$, and $\beta_2 = 3/2$. Pearson's criterion K, for determining the type of density function in terms of its parameters, is given by

$$K = \frac{\beta_1(\beta_2 + 3)^2}{4(4\beta_2 - 3\beta_1)(2\beta_2 - 3\beta_1 - 6)} \tag{17}$$

which yields $K = 0$ for $f(y)$. This, when considered in conjunction with β_1 and β_2, indicates that $f(y)$ corresponds to the Type II density function of the Pearson system.

From a method provided by Tchebycheff, and discussed by Church[6] and Cramer,[7] the distribution of the means of samples of a small number of items n, selected at random from a parent population can be determined with reasonable accuracy, even for populations whose distributions are slightly skew, by relating the moments and parameters of the distribution of the means of the samples to those of the sampled population in the following manner:

$$\begin{aligned} &\text{(a) } M_1 = \mu_1 && \text{(e) } B_1 = \frac{\beta_1}{n} \\ &\text{(b) } M_2 = \frac{\mu_2}{n} && \text{(f) } B_2 = 3 + \frac{(\beta_2 - 3)}{n} \\ &\text{(c) } M_3 = \frac{\mu_3}{n^2} && \\ &\text{(d) } M_4 = \frac{\mu_4}{n^3} + \left[\frac{3(n-1)}{n^3}\right][\mu_2^2] && \end{aligned} \tag{18}$$

where M_1, M_2, M_3, M_4, B_1 and B_2 are the moments and parameters of the distribution of means of samples of n and μ_1, μ_2, μ_3, μ_4, β_1, and β_2 represent these quantities for the parent population. Church has also shown that if the parent population is distributed as a Type I, II, III, IV or VII distribution of the Pearson system, with parameters of moderate size, the distribution of the means of the samples and parent population will be of the same type.

Having previously shown that the density function $f(y)$ determined from the $E \cos \theta$ population corresponds to the Type II density function of the Pearson system, it follows from Church's work that the distribution of the means of samples from such a population corresponds to the Type II distribution.

Once again we consider (7), where the instantaneous value E_n at any time t is

$$E_n = E \sum_{k=1}^{n} \cos \theta_k .$$

The mathematical expectation or mean of $\cos \theta$ is then

$$\mathcal{E}(\cos \theta) = \frac{1}{n} \sum_{k=1}^{n} \cos \theta_k = \frac{E_n}{nE}. \tag{19}$$

[6] A. E. R. Church, *Biometrika*, vol. 18, p. 321; 1926.

[7] H. Cramer, "Mathematical Method of Statistics," Princeton University Press, Princeton, N. J., p. 345; 1950.

Thus, letting $x = E_n/nE$, the density function of the distribution of the means of samples of $\cos \theta$ is

$$f(x) = \frac{1}{aB(1/2, m+1)} \left(1 - \frac{x^2}{a^2}\right)^m, \quad -a \le x \le a, \tag{20}$$

which is the Pearson Type II density function, where

$$\begin{aligned} a^2 &= \frac{2B_2 M_2}{(3 - B_2)}, && a^2 \ge 1.5, \\ m &= \frac{(5B_2 - 9)}{2(3 - B_2)}, && 1.50 \le m \le 21.50, \end{aligned} \tag{21}$$

for $2 \le n \le 12$. $B(1/2, m + 1)$ represents the complete Beta function. The distribution function $F(x)$ is given as

$$F(x) = \frac{2}{aB(1/2, m+1)} \int_0^x (1 - t^2/a^2)^m \, dt. \tag{22}$$

Letting $u = t^2/a^2$

$$\begin{aligned} F(x) &= \frac{1}{B(1/2, m+1)} \int_0^{(x/a)^2} u^{-1/2}(1-u)^m \, du \\ &= I_{(x/a)^2}(1/2, m+1), \end{aligned} \tag{23}$$

the incomplete Beta function.

Thus, the probability of exceeding any particular value of the mean of $\cos \theta$ is $1 - F(x)$.

From (23), then

$$1 - F(x) = \left[1 - I_{(x/a)^2}(p, q)\right] \tag{24}$$

where $p = 1/2$, $q = m + 1$.

This expression is readily evaluated since the values of the function, $I_{(x/a)^2}(p, q)$ have been compiled and published in tabular form.[8] The resulting probability curves are shown in Fig. 2.

Experimental Procedure

Laboratory equipment suitable for measuring the amplitude distribution of complex signals resulting from combination of $n = 2, 3, 4, 5, 6$ sine waves of different audio frequencies has been developed for the purpose of checking the validity of the mathematical analysis. Fig. 1 (opposite) is a block diagram of the test equipment. The assumptions in the mathematical analysis regarding randomness of phase should be fulfilled experimentally since the sinusoidal signals are derived from independent sources and all values of initial phase angle are equally probable for each oscillator.

To correlate the experimental considerations with the theoretical results it is necessary to determine experimentally the probability that the instantaneous voltage of the complex signal exceeds some arbitrary value R. This may be reduced to a measurement of the percentage of time the instantaneous voltage exceeds R, providing the measurements are continued for a time long enough to give statistical meaning to the data.

[8] Karl Pearson, "Tables of the Incomplete Beta Function," University Press; 1934.

Referring to Fig. 1, the equal amplitude outputs of the desired number of oscillators are linearly summed in the mixer to give a complex resultant signal which is amplified and applied to the Schmitt trigger input. The Schmitt trigger is adjusted to operate at the input voltage level R and to remain in operation until the input signal drops below R. During this time a positive dc square wave available at the output of the Schmitt trigger enables the gate following, thus allowing the timing signals from the standard frequency source to pass through the gate and to be counted by the counter.

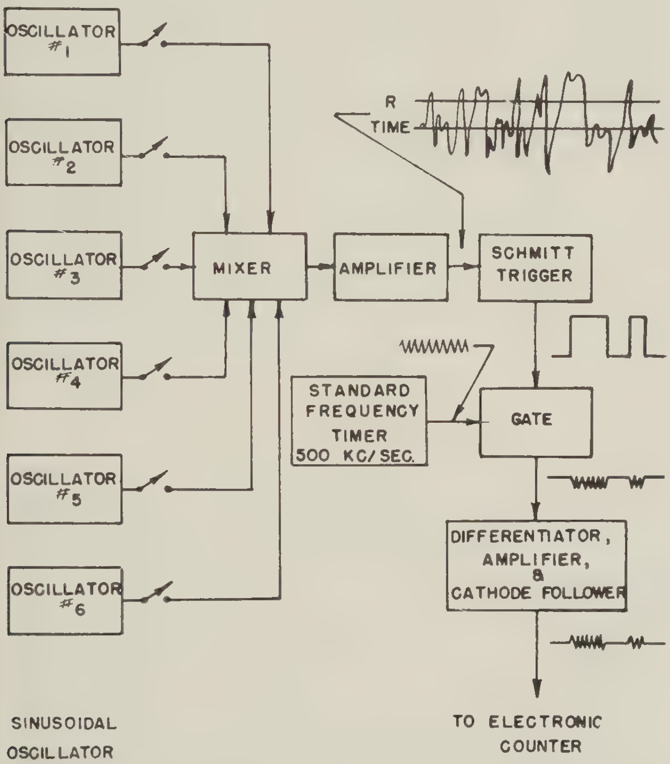


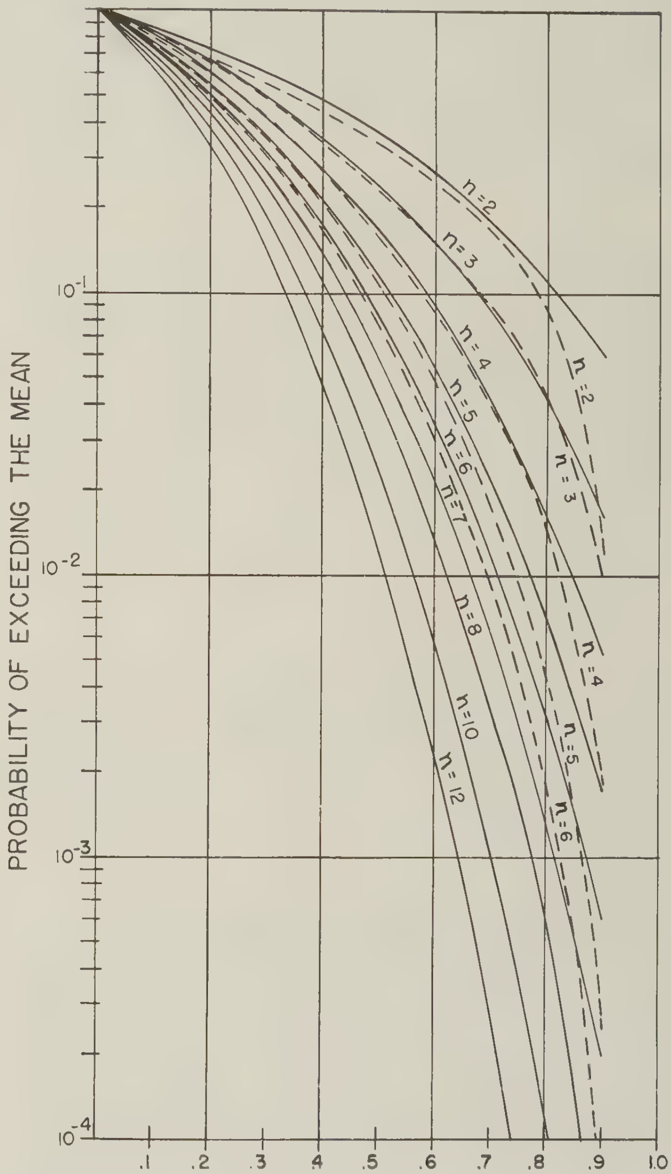
Fig. 1—Block diagram of equipment employed for determination of the distribution of amplitude with time of the complex wave resulting from linear summation of a number of sinusoidal signals.

The percentage of time the instantaneous voltage of the resultant signal exceeds R may be determined by the ratio of the number of counts observed to the total number of counts possible during the time of observation. If the experiment is repeated for different values of R, it is then possible to plot a curve of the percentage of time the instantaneous voltage exceeds R as a function of R.

The timer and counter must each be capable of operation at rates sufficiently high to give an accurate measure of the duration of each excursion of the complex signal above the value R. This is critical when R is large and thus the excursions above R necessarily of short duration.

The measurements were made in the above described manner and the results are plotted in Fig. 2. In determining the percentage of time a particular value of R is exceeded, the mean value of fifteen observations over periods of one second each was used.

It is of interest to note that frequency modulating the individual oscillators in a manner suitable for communicating information in a binary form did not yield results significantly different from those plotted in Fig. 2. Thus it appears valid to employ the results of this analysis in determining allowable modulation levels for an FM-AM transmission system.



MEAN VALUE X OF A SAMPLE OF n SINUSOIDS WHERE $X = \left|\frac{E_n}{nE}\right|$

Fig. 2—Distribution curves of n sinusoids combined at random.
——— Theoretical
- - - - - Experimental

Discussion of Results

The methods of Pearson and Tchebycheff, in addition to other well-known methods, were applied to a similar theoretical analysis by Margaret Slack.[9] However, it was stated in Miss Slack's treatment that representation of

[9] M. Slack, *Jour. IEE*, vol. 93, p. 76; 1946.

TABLE I

Modulation Indices Calculated According to the Various Methods Discussed

	(A)	(B)	(C)	(D)	(E)	(F)	(G)	(H)
		Assumed Normal Distribution, $M = \gamma^{-1}(2/n)^{1/2}$				Distributions of Fig. 2, $M = 1/xn$		
n	Linear Summation (Landon) $M = 1/n$	No Overmodulation (Landon RMS) $\gamma = 4$	Overmodulation Time 1% $\gamma = 2.575$	Overmodulation Time 6% $\gamma = 1.88$	Overmodulation Time 15% $\gamma = 1.44$	Overmodulation Time 1%	Overmodulation Time 6%	Overmodulation Time 15%
2	.500	.249	.388	.532	.694	—	.556	.676
3	.333	.204	.317	.435	.568	—	.441	.556
4	.250	.177	.275	.376	.491	.296	.382	.481
5	.200	.158	.245	.336	.438	.260	.341	.435
6	.167	.144	.224	.307	.401	.235	.309	.392
8	.125	.125	.194	.266	.347	.200	.266	.344
10	.100	.112	.174	.238	.311	.178	.238	.308
12	.083	.102	.159	.217	.284	.162	.219	.282

the true distributions by distributions resulting from evaluation of the Type II density function of the Pearson system was inaccurate for the cases of $n = 3$ and $n = 4$. A comparison of the results obtained here, using the Pearson Type II density function, with those obtained by Miss Slack employing another method, indicates that such a conclusion is unwarranted.

In addition, it was found that Miss Slack's values, obtained by the use of the Pearson system for $6 \leq n \leq 10$, appear to be somewhat higher throughout the entire range of the probability domain than those obtained in this analysis. A similar disagreement with the results for the region of maximum amplitude (small probabilities) obtained by Miss Slack from the normal distribution for the case where $n = 10$, has been mentioned in an analysis performed by Bennett,[10] utilizing a different method.

It is also of interest to note that the values obtained in this analysis by use of the Pearson system for the case of $n = 10$, are in good agreement with those obtained by Bennett.

The theoretical and experimental distributions as plotted in Fig. 2 are in close agreement for probabilities greater than 0.01, with the exception of $n = 2$. Since experiment shows the distributions to be essentially unchanged when the sinusoids are frequency modulated, it is valid to determine the allowable degree M to which each subcarrier may amplitude modulate the final carrier in an FM-AM frequency multiplexed system according to these distributions. The data in Table I compare modulation levels calculated according to the various methods discussed, for different degrees of overmodulation.

[10] W. R. Bennett, *Quart. Appl. Math.*, vol. 5, p. 385; 1948.

From Table I it is observed that for $n \geq 8$ the modulation levels M determined with the normal distribution and the distributions of this paper are in two figure agreement for the three overmodulation times considered. Thus in practice the normal distribution may be assumed, when overmodulation is allowed, in the determination of M for $n \geq 8$. If overmodulation is not desired, then the normal distribution may still be assumed, as in column (B) for $n \geq 8$, but linear summation as in column (A) must be employed for $n < 8$. Use of the linear summation of column (A) for $n \geq 8$ would also assure no overmodulation but would result in inefficient use of the system because of unnecessarily low modulation levels.

For $n < 8$, if overmodulation time is not to exceed one per cent, then comparison of columns (C) and (F) shows the error in assuming the normal distribution to be appreciable for small n but to decrease with increasing n. If larger overmodulation times are to be tolerated, then a comparison between columns (D) and (G) and (E) and (H) shows the error to be somewhat smaller for all $n < 8$, but again tending to be greatest for small n.

That approximating with a normal distribution leads to errors generally dependent on the overmodulation times allowed, is a consequence of the fact that the distributions found here are approximated by the normal distribution better for some probability regions than for others, the regions of low probability showing the greatest deviation. Because of this and since the normal distribution becomes generally a poorer fit as n decreases below 8, it is advisable to determine allowable values of M according to the distributions presented here for $n < 8$.

The calculations determining the data in Table I for the cases where overmodulation is allowed are based on the assumption that overmodulation occurs whenever the complex modulating signal exceeds the carrier level in either direction from the reference zero of the modulating signal; that is, both tails of the probability distribution have been included. This is not strictly correct in the usual case. Overmodulation certainly occurs when the carrier is reduced to zero but on the positive swings of the modulating signal, the point at which overmodulation occurs is not so clearly defined since it will depend in part on the saturation characteristics of the modulated amplifiers. Conceivably it is possible, and perhaps advisable in certain situations, to design the modulation stages to handle, with low distortion, positive swings of the signal greater than the unmodulated carrier level. Under these conditions overmodulation time as shown in Table I will be reduced nearly half for a particular value of M. Stated differently, M can be increased somewhat for each overmodulation time shown in the table.

Finally, the overmodulation times considered were arbitrarily chosen and it is evident that M can be accurately determined for any desired degree of overmodulation in excess of one per cent from the distributions of Fig. 2. The degree of overmodulation permissible in any situation will depend in part upon the noise inherent in the electrical equipment and surrounding environment. The best choice of overmodulation for a particular transmission system will be further dependent upon the number of subcarriers employed and the character of noise resulting from overmodulation. This essentially reduces to a determination of the resulting loss of information as a function of the degree of overmodulation, a topic for further study.

Acknowledgment

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On the Response of a Certain Class of Systems to Random Inputs*

JACK HEILFRON†

Summary—**This paper deals with the connection between vector Markoff processes and the response of a lumped constant parameter linear system composed of a finite number of elements. It was known that if a Gaussian process which is one component of a vector Markoff process passes through such a system, the result is also Gaussian and may be considered as one component of a higher dimensional vector Markoff process. We show that the term Gaussian may be excluded in the above statement. The practical importance of this result is that if one can determine the initial and transition probabilities of this vector Markoff process, one can also determine the complete statistical properties of the output of the system. This further implies that the determination of the properties of the output for the class of not necessarily Gaussian inputs mentioned above is not as difficult as might be expected from the results available for just the first probability distribution for non-Gaussian inputs.**

Introduction

During the last decade there has been considerable interest on the part of engineers and physicists in problems dealing with random or stochastic processes. One such problem concerns the determination of the statistical properties of the output of a system, such as shown in Fig. 1, whose input is a known random process. In particular, the system is composed of a nonlinear, no memory device commonly called a detector followed by a linear filter. This system could correspond, for example, to the final detector—video amplifier portion of a communication or radar receiver. Knowledge of the statistical properties (i.e., finite dimensional probabilities) of the output of such a receiver is required for any complete theory of signal detection in presence of noise.[1]

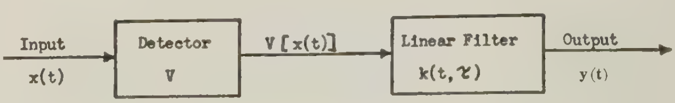

Fig. 1

As far as available results are concerned, it is well known that if the input is a Gaussian process and the system is linear (i.e., the detector degenerates into a linear device), then the output is also Gaussian and its mean and correlation function can be readily computed. These two functions completely describe a Gaussian

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[1] R. C. Davis, "Detectability of random signals in the presence of noise," *Trans. IRE*, vol. IT 3, pp. 52–62; March, 1954.

process.[2] For other situations, the problem is much more difficult. Kac and Siegert[3] have given a method for determining the first-order probability distribution of the output provided the input is Gaussian and the nonlinear device is a square law detector. Siegert[4] and Fortet[5] independently derived equations for a certain conditional probability distribution of the output for the cases in which the input is one component of a vector Markoff process. As we shall show, this conditional probability distribution is all that is required to describe the output, i.e., to obtain all of its finite dimensional probabilities, in the event that the linear filter of Fig. 1 is a simple low-pass filter.

Actually both Siegert and Fortet investigated functionals of the form

$$u(t) = \int_0^t h(\tau) V[x(\tau)]\, d\tau, \tag{1}$$

which does not correspond to the output of the system shown in Fig. 1. If the filter is a simple low-pass filter then its impulse response is

$$k(t - \tau) = \beta e^{-\alpha(t-\tau)}, \tag{2}$$

and the system output is

$$y(t) = \int_0^t \beta e^{-\alpha(t-\tau)} V[x(\tau)]\, d\tau = e^{-\alpha t} \int_0^t \beta e^{\alpha\tau} V[x(\tau)]\, d\tau. \tag{3}$$

Thus,

$$u(t) = e^{+\alpha t} y(t) = \int_0^t \beta e^{\alpha\tau} V[x(\tau)]\, d\tau \tag{4}$$

is of the form suitable for the application of Siegert's and Fortet's results. For more general filters, the system output may not be so easily related to a functional of the proper form.

We shall show just what probabilities are actually required for the more general filter case by showing the connection between the output of the system and a certain vector Markoff process. Extensions of Siegert's and Fortet's results and also the Kac-Siegert method have been made which will yield, in principle, the required probabilities.[6] To avoid undue complications let us closely examine the simplest special case, then generalize.

Special Case

Let the input to the system be a Markoff process and the linear device be a simple lowpass filter. The output of the system is

$$y(t) = \int_0^t \beta e^{-\alpha(t-\tau)} V[x(\tau)]\, d\tau. \tag{5}$$

If $t_1 < t_2 < \cdots < t_n$,

$$y(t_i) = \int_0^{t_i} \beta e^{-\alpha(t_i-\tau)} V[x(\tau)]\, d\tau$$
$$= \sum_{j=1}^{i} e^{-\alpha(t_i-t_j)} Z(t_j), \tag{6}$$

where

$$Z(t_j) = \int_{t_{j-1}}^{t_j} \beta e^{-\alpha(t_j-\tau)} V[x(\tau)]\, d\tau \tag{7}$$
$$= y(t_j) - e^{-\alpha(t_j-t_{j-1})} y(t_{j-1}). \tag{8}$$

The joint probability density[7] of the output at the times $t_1, \cdots, t_n$ defined as

$$p_y(y_1; t_1, \cdots, y_n; t_n)$$

is related to that of the increments, defined as

$$p_z(Z_1; t_1, \cdots, Z_n; t_n),$$

by the equation

$$p_y(y_1; t_1, \cdots, y_n; t_n) = p_z(Z_1; t_1, \cdots, Z_n; t_n), \tag{9}$$

provided that [from (8)]

$$Z_j = y_j - e^{-\alpha(t_j-t_{j-1})} y_{j-1}. \tag{10}$$

Thus, to determine the finite dimensional probabilities of the output, we may confine our attention to the corresponding ones for the increments. Also by considering the values of the input at times $0, t_1, \cdots, t_n$ and the joint density function

$$p(x_0; 0, Z_1, x_1; t_1, \cdots, Z_n, x_n; t_n), \tag{11}$$

we have

$$p(Z_1; t_1, \cdots, Z_n; t_n)$$
$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p(x_0; 0, Z_1, x_1; t_1, \cdots,$$
$$Z_n, x_n; t_n)\, dx_0 \cdots dx_n. \tag{12}$$

Thus we may also consider the joint density function of the input and the increments given by (11). Expanding this density function in terms of conditional probabilities gives

$$p(x_0; 0, Z_1, x_1; t_1, \cdots, Z_n, x_n; t_n)$$
$$= p(x_0; 0) p(Z_1, x_1; t_1 \mid x_0; 0) \cdots$$
$$p(Z_n, x_n; t_n \mid x_0; 0, Z_1, x_1; t_1, \cdots, Z_{n-1}, x_{n-1}; t_{n-1}). \tag{13}$$

By examining a typical term, namely the joint density function of $Z(t_j)$ and $x(t_j)$ on the hypothesis that $x(t_i) = x_i$, $i = 0, \cdots, n$ and $Z(t_i) = Z_i$, $i = 1, \cdots, n$, we see

[2] H. Cramer, "Mathematical Methods of Statistics," Princeton University Press, Princeton, N. J.; 1951.

[3] M. Kac and A. J. F. Siegert, "On the theory of noise in radio receivers with square law detectors," *Jour. Appl. Phys.*, vol. 18, pp. 383-397; 1947.

[4] A. J. F. Siegert, "Passage of stationary processes through linear and nonlinear devices," *Trans. IRE*, vol. IT 3, pp. 4–25; March, 1954.

[5] R. Fortet, "Additive functionals of a Markoff process," *Ann. Math.*, to be published.

[6] J. Heilfron, "On the Response of Linear Systems to Non-Gaussian Noise," Ph.D. Dissertation, Dept. Engrg., Univ. of Calif., Los Angeles, Calif.; 1954.

[7] To avoid mathematical details, we shall assume that all density functions mentioned exist.

that

$$p(Z_j, x_j; t_j \mid x_0; 0, Z_1, x_1; t_1, \cdots, Z_{j-1}, x_{j-1}; t_{j-1}) = p(Z_j, x_j; t_j \mid x_{j-1}; t_{j-1}), \qquad (14)$$

because of the definition of the increments, (7), and the fact that the input is a Markoff process. Thus

$$p(x_0; 0, Z_1, x_1; t_1, \cdots, Z_n, x_n; t_n) = p(x_0; 0) \prod_{j=1}^{n} p(Z_j, x_j; t_j \mid x_{j-1}; t_{j-1}), \qquad (15)$$

which states that $\{Z(t_j), x(t_j)\}$ is a two-dimensional Markoff process. It is of a degenerate type, since the transition probability

$$p(Z_j, x_j; t_j \mid x_{j-1}; t_{j-1}) \qquad (16)$$

is independent of Z_{j-1}. Since we have assumed a knowledge of the input process, $p(x_0; 0)$ is known. Therefore, one need only determine the transition probabilities of the vector Markoff process to obtain all of its finite dimensional probabilities. These in turn give the same for just the increments by (12) and thence of the output by (9) and (10).

The transition probability of the above mentioned vector Markoff process is the function that, in effect, both Siegert and Fortet consider (note comments in the Introduction). It can readily be shown that the process $\{y(t), x(t)\}$ is also a Markoff process but its transition probabilities are more difficult to work with.

General Situation

Let us now generalize to the case in which the input, which we denote by $x^1(t)$, is one component of a vector Markoff process

$$X(t) = \{x^1(t), \cdots, x^N(t)\}$$

and the filter contains a finite number of lumped, constant value elements. In this event, the filter's impulse response is

$$k(t - \tau) = \sum_{l=1}^{M} \beta_l e^{-\alpha_l (t-\tau)}. \qquad (17)$$

Thus, the output is (see Fig. 2)

$$\begin{aligned} y(t) &= \int_0^t k(t - \tau) V[x(\tau)]\, d\tau \\ &= \sum_{l=1}^{M} \int_0^t \beta_l e^{-\alpha_l (t-\tau)} V[x'(\tau)]\, d\tau \\ &= \sum_{l=1}^{M} y^l(t), \end{aligned} \qquad (18)$$

where

$$y^l(t) = \int_0^t \beta_l e^{-\alpha_l (t-\tau)} V[x^1(\tau)]\, d\tau; \quad l = 1, \cdots, M. \qquad (19)$$

By considering all of the $y^l(t)$'s and the vector $X(t)$ as was done for the special case before, we find that the $M + N$ dimensional vector

$$\{y^1(t), y^2(t), \cdots, y^M(t), x^1(t), \cdots, x^N(t)\} \qquad (20)$$

is a Markoff process. Also, a linear transformation of a Markoff process results in a Markoff process so that

$$\{y(t), y^2(t), \cdots, y^M(t), x^1(t), \cdots, x^N(t)\} \qquad (21)$$

is also an $M + N$ dimensional Markoff process and the output is one component of it (namely the first as we have written it). Its initial probabilities are known so one need only determine its transition probabilities. Equations for these probabilities have been derived.[6] Actually it was more convenient to work with increments as in the special case considered here.

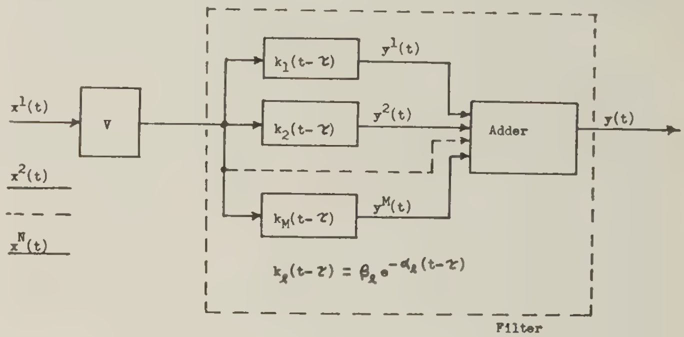

Fig. 2

Conclusion

We have shown that if the input to the system of Fig. 1 is one component of a vector Markoff process and the linear filter contains a finite number of lumped, constant elements, then the output is also one component of a (higher dimensional) vector Markoff process. The dimensionality of the "output" Markoff process equals the sum of the dimensionality of the "input" process and the number of natural frequencies of the filter.

As a final remark, it is known[8,9] that a Gaussian process with a rational spectral density (power spectrum) may be considered as one component of a vector Markoff process and thus is a suitable input. Also the output of the system we have considered is a suitable input for another system of the same type.

[8] J. L. Doob, "The elementary Gaussian processes," *Ann. Math. Stat.*, vol. 15, pp. 229–282; 1944.

[9] M. C. Wang and G. E. Uhlenbeck, "On the theory of Brownian motion—II," *Rev. Mod. Phys.*, vol. 17, pp. 323–342; 1945.

Noise in Driven Systems*

J. M. RICHARDSON†

Summary—**It is known that a direct relation exists between the noise in a system in equilibrium and transient drift toward equilibrium. It seems that a similar relation should exist for a system in a nonequilibrium stationary state. It is now necessary to distinguish between two types of transients; those produced by selecting those systems satisfying certain initial and those produced by actual physical perturbation. It is shown that a simple relation exists between noise and the transients produced by selection and that no relation exists in the case of transients produced by perturbation. In the equilibrium case it is shown that the two types of transients, though still logically and operationally distinct, can be described by the same impedance operator.**

I. Introduction

Many years ago H. Nyquist[1] derived a simple formula connecting the noise in system in equilibrium with the transient response of the system. It is our purpose here to see to what extent his results can be carried over to a system in a nonequilibrium stationary state. By the latter we mean a system is more or less strongly driven by some constant force (applied by a second system, which itself cannot be stationary) in such a way that is statistically (i.e., macroscopically) stationary while not in equilibrium. We call such a system a driven system. A good example might be a resistor or a germanium diode connected to a constant voltage source. It is known that strongly driven devices exhibit noise differing in magnitude and quality from ordinary thermal noise. It would be useful and interesting if the noise in such systems could be related to the transient behavior.

There are two distinct means for producing transients in a system. The first is by selecting from an ensemble those systems satisfying certain initial conditions. Or alternatively, by selecting the starting times of observation on a given system by the same criterion. We call this type a transient produced by selection. It should be emphasized that here no physical action has been directed against the system. The observer has only made a decision about what systems to observe or alternatively when to start observing a given system.

The second type of transient is produced by applying a transient force to the system. We call this type a transient produced by perturbation.

We will show that a simple relation between noise and the transients exists when the latter are of the first type and that no such relation exists when they are of the second; and further that, when the system's normal state is an equilibrium, one of the two types of transients can be described by the same impedance operator. It follows that noise is simply related to both types of transients.

* Presented at the Western Electronics Show and Convention, Los Angeles, Calif., August 25–27, 1954.
† Ramo-Wooldridge Corporation, Los Angeles 45, Calif.
[1] H. Nyquist, *Phys. Rev.*, vol. 32, pp. 110; 1928.

II. Noise and Transients Produced by Selection

Here we investigate the relation between noise as defined by the autocorrelation function or power spectrum and the average transient behavior following a selection process. The normal state of the system is stationary, but not equilibrium.

For the sake of simplicity we shall treat the problem in the classical approximation. A quantum approach introduces interesting additional difficulties; however, these are not closely relevant to the questions that concern us here. Let our information concerning the driven system be given by an observable α, which is a given function of the microscopic state of the system, that is, a function of coordinates and momenta. As an example, we might consider the driven system to be a resistor and the observable α to be the electric current flowing through it.

To formulate the equation of motion for α, we are obliged to consider the whole isolated system. This means that we must consider the driven system in conjunction with the system that maintains it in a nonequilibrium stationary state. Let us call the latter the dc driving system. It is not possible for the combined system to be as a whole in a nonequilibrium stationary state. In order to maintain the driven system in a nonequilibrium stationary state (at least for a finite interval of time) the dc driving system cannot itself be in a stationary state. Consider a resistor as an example of a driven system and a battery as an example of a dc driving system (with appropriate electrical connections, of course). In order to maintain a constant voltage across the resistor for a certain length of time, the battery will suffer internal chemical changes. In short, we must use in the equations of motion the Hamiltonian H for the combined system: driven system plus dc driving system. Furthermore, the distribution function describing the statistical situation cannot be stationary in the strict sense (except for the special case of equilibrium). However, it can in effect be stationary in the more limited sense that mean values of functions of the microscopic state of the driven systems are constant, or very nearly so, during a time interval of sufficiently long duration.

Returning now to the formulation of the equation of motion for the observable α, the time rate of change of α is given by the equation

$$\frac{d\alpha}{dt} \equiv \dot{\alpha} = [\alpha, H] \equiv \sum_{S=1}^{N} \left(\frac{\partial \alpha}{\partial q_s} \frac{\partial H}{\partial p_s} = \frac{\partial d}{\partial p_s} \frac{\partial H}{\partial q_s} \right)$$

$$\equiv \mathfrak{L}\alpha, \tag{1}$$

in which the operator $\mathfrak{L}$ is an abbreviation for the operation $[(\ ), H]$. The solution of the equation of motion can be expressed in the form

$$\alpha_t = e^{t\mathfrak{L}}\alpha_0, \qquad \alpha_0 \equiv \alpha, \tag{2}$$

The subscript t means that the quantity to which it is affixed is expressed in terms of the co-ordinates and momenta of the system a time interval t earlier. The subscript 0 implies that the function is expressed in terms of the co-ordinates and momenta of the same instant. In accordance with usual conventions the absence of any subscript at all will be taken to mean the same thing.

In the present treatment, we will approach statistical questions from the standpoint of a statistical ensemble rather than a time average standpoint. According to previous statements this ensemble can be stationary (when not in equilibrium) only in the limited sense that the driven system appears stationary. Let the carets $\langle\ \rangle$ devote an average in such an ensemble. We find first that the mean value of the observable α, giving our information concerning the driven system, is constant in time:

$$\langle\alpha_t\rangle = \langle\alpha\rangle = a^0. \tag{3}$$

The average value of α at time t, given that $\alpha = x$ at time 0, is

$$a(t; x) = \langle\alpha_t\delta(\alpha - x)\rangle / \langle\delta(\alpha - x)\rangle. \tag{4}$$

The function $\delta(a - x)$ is the Dirac delta-function of $\alpha - x$ and it performs the task of selecting those members of the statistical ensemble that satisfy the initial condition $\alpha = x$.

Defining

$$\Delta a(t; x) = a(t; x) - a^0,$$
$$\Delta x = x - a^0, \tag{5}$$

we obtain the following identities

$$\int dx\ \Delta a(t; x)\langle\delta(\alpha - x)\rangle = 0,$$
$$\int dx\ \Delta x\langle\delta(\alpha - x)\rangle = 0. \tag{6}$$

The first expresses the fact that an average over all the possible initial conditions is equivalent to no initial condition at all. We obtain by simple manipulations the further results

$$\int dx\ \Delta a(t; x)\Delta x\langle\delta(\alpha - x)\rangle = \langle\Delta\alpha_t\Delta\alpha\rangle$$
$$\int dx\ (\Delta x)^2\langle\delta(\alpha - x)\rangle = \langle(\Delta\alpha)^2\rangle, \tag{7}$$

in which

$$\Delta\alpha_t = \alpha_t - a^0. \tag{8}$$

Let us pause to consider the meaning of $\Delta a(t; x)$ and various relations involving it. As stated in the introduction, there are two distinct ways of producing a transient response in a system: by selection and by perturbation. The function $\Delta a(t; x)$ is the response produced by selection, the basis of selection being $\alpha = x$ at time 0. An important point here is that nothing physical is done to any system. It is just a decision by the observer of which systems to observe[2]. If one wishes to assume a time average in preference to an ensemble average viewpoint one may make the following interpretation of $\Delta a(t; x)$. Suppose now we have one system, not a statistical ensemble, and that we have a device which starts recording the time history of α every time $\alpha = x$. The average of such recordings is $a(t; x) = a^0 + \Delta a(t; x)$, where now t is not the absolute time, but the time elapsed since the start of each recording.

The first part of (7) establishes a connection between the transient produced by selection, $\Delta a(t; x)$, and the statistical properties of the entire ensemble embodied in the autocorrelation function $\langle\Delta\alpha_t\,\Delta\alpha\rangle$. The autocorrelation function describes the noise in α exhibited by each system in the ensemble. The power spectrum is given in terms of the autocorrelation function by the well-known Wiener-Khinchin Theorem.

A simple relation between noise and transient response can be obtained by expanding $\Delta a(t; x)$ in powers of Δx and neglecting quadratic and higher order terms:

$$\Delta a(t; x) = g(t)\Delta x + 0((\Delta x)^2). \tag{9}$$

Insertion of this result into the first part of (7) yields with help of the second (7) the result

$$\begin{aligned}\langle\Delta\alpha_t\Delta\alpha\rangle &= \int dx\ \Delta a(t; x)\Delta x\langle\delta(\alpha - x)\rangle \\ &= g(t)\int dx\ (\Delta x)^2\langle\delta(\alpha - x)\rangle \\ &= g(t)\langle(\Delta\alpha)^2\rangle.\end{aligned} \tag{10}$$

Thus the shape of the autocorrelation function is given by $g(t)$ describing the transient response produced by selection.

Since

$$\begin{aligned}\langle\Delta\alpha_t\Delta\alpha\rangle &= \langle\Delta\alpha\Delta\alpha_{-t}\rangle \\ &= \langle\Delta\alpha_{-t}\Delta\alpha\rangle, \\ g(t) &= g(-t).\end{aligned} \tag{11}$$

It follows from (10) that $g(t)$ is an even function of time. It must be pointed out that this is not the principle of microscopic reversibility. If we were dealing with a set of observables so that α would be regarded as a vector and $g(t)$ as a matrix, the principle of microscopic reversibility would have to do with the symmetry properties of the matrix g. In particular, if the normal macroscopic state toward which transients tended were an equilibrium state (not merely a stationary state, in the limited sense) and if the observables were all odd or even, the matrix would be symmetrical. But in the present case of nonequilibrium stationarity, no symmetry properties would exist.[3]

[2] Since we are working in the classical approximation, the physical perturbation of a system by the process of observation can be arbitrarily small.

[3] Except, of course, for the matrix equivalent of the statement that the scalar $g(t)$ is an even function of time: i.e., that the matrix $g(-t)$ is equal to the transpose of the matrix $g(+t)$.

III. Modified Nyquist Theorem

It is desirable to transform the results of the previous section into a form more closely related to the usual formalism of circuit theory. To this end we must define an impedance operator $Z(d/dt)$ giving g as the response to an impulse (δ-function) input, even though no such input exists physically. We define such an impedance operator by the equation

$$Z\left(\frac{d}{dt}\right)[1(t-t')g(t-t')] = c\delta(t-t'). \tag{12}$$

The quantity c is an arbitrary constant to be assigned any value we choose depending upon later circumstances. The quantity $1(t - t')$ is unit step function defined by

$$\begin{aligned} 1(t-t') &= 0, \qquad t < t' \\ &= 1, \qquad t > t'. \end{aligned} \tag{13}$$

Thus, according to (12), $1(t - t')g(t - t')$ is the response of the system to the input $c\delta(t-t')$. Multiplying the above response and input by Δx and using (9) we arrive at the alternative statement that $1(t)\Delta\ a(t; x)$ is the response to the input $c\Delta x\delta(t)$. It should be emphasized again that these inputs are physically fictitious and are introduced in a formal way for the purpose of transforming the previous results into a form more closely related to conventional circuit theory.

The impedance operator can be shown to be of the form

$$c^{-1}Z\left(\frac{d}{dt}\right) = \frac{d}{dt} - g'(+0) + \int_0^\infty du\, K(u)e^{-u\frac{d}{dt}}$$

$$Z\left(\frac{d}{dt}\right)1(t)\Delta a(t; x) = c\Delta x\delta(t), \tag{14}$$

in which $K(u)$ may be determined readily from (12) by the use of the Laplace transform. $K(u)$ may be interpreted as the factor coupling the present rate of change of $\Delta a(t; x)$ with the past value $\Delta a(t - u; x)$. If the system in question is a nearly perfect resistor in that its frequency response is nearly flat at high frequencies and absolutely flat at low frequencies the function $K(u)$ will attenuate very rapidly with increasing u.

We introduce now the concept of random driving force. This is a fictitious input imagined to "cause" the fluctuations of α about its mean value. We define the random force ϵ_t by the relation

$$\epsilon_t = Z\left(\frac{d}{dt}\right)\Delta\alpha_t\,. \tag{15}$$

We must now find the autocorrelation function for this random force. By straightforward manipulations involving the use of (12) we obtain

$$\begin{aligned} \langle\epsilon_t\epsilon_{t'}\rangle \equiv C(t-t') &= Z\left(\frac{d}{dt}\right)Z\left(\frac{d}{dt'}\right)\langle\Delta\alpha_t\Delta\alpha_{t'}\rangle \\ &= Z\left(\frac{d}{dt}\right)Z\left(\frac{d}{dt'}\right)g(t-t')\langle(\Delta\alpha)^2\rangle \\ &= Z\left(\frac{d}{dt}\right)Z\left(\frac{d}{dt'}\right)[1(t-t')g(t-t') \\ &\qquad + 1(t'-t)g(t'-t)]\langle(\Delta\alpha)^2\rangle \\ &= \left[Z\left(\frac{d}{dt'}\right)\delta(t-t') + Z\left(\frac{d}{dt}\right)\delta(t'-t)\right]c\langle(\Delta\alpha)^2\rangle \\ &= \left\{\left[Z\left(\frac{-d}{dt}\right) + Z\left(\frac{d}{dt}\right)\right]\delta(t-t')\right\}c\langle(\Delta\alpha)^2\rangle. \end{aligned} \tag{16}$$

This is the modified Nyquist theorem in the time domain. It connects the autocorrelation function of ϵ_t, $C(t - t') = \langle\epsilon_t\epsilon_{t'}\rangle$, with $Z(-d/dt) + Z(d/dt)$, the resistive part of impedance expressed in the time domain.

It is desirable to exhibit $C(t)$ in alternative forms. By introducing the explicit expression (14) for $Z(d/dt)$ we obtain from (16) the result

$$C(t) = [-2g'(+0)\,\delta(t) + 1(t)K(t) + 1(-t)K(-t)]c^2\langle(\Delta\alpha)^2\rangle. \tag{17}$$

By taking the Fourier transform of (16) we obtain the modified Nyquist theorem in the frequency domain:

$$\begin{aligned} \mathcal{C}(\omega) &= \int_{-\infty}^{+\infty} dt\, c^{-i\omega t}C(t) = 2\,\mathrm{Re}\, Z(i\omega)c\langle(\Delta\alpha)^2\rangle \\ &= \pi P(\omega). \end{aligned} \tag{18}$$

Here $P(w)$ is the power spectrum as conventionally defined with the frequency ω expressed in radians per second.

To show that (18) reduces to the usual statement of the Nyquist theorem, let the statistical ensemble be in equilibrium instead of merely stationary, and further, let the observable α be the electric current and let c be the inductance or whatever is the coefficient of d/dt in the impedance giving the response of the current to a voltage input. We then obtain

$$P(\omega) = \frac{1}{\pi}\,\mathcal{C}(w) = \frac{2}{\pi}\,\mathrm{Re}\, Z(i\omega)kT, \tag{19}$$

where k is the Boltzmann constant and T is the absolute temperature.

IV. Response Produced by a Perturbation

We wish now to investigate the response of the driven system to an actual physical perturbation. More specifically, we must consider the interaction between a transient driving system and the previous driven system combined with the dc driving system. We consider a statistical ensemble in which the driven system combined with the dc driving system is initially stationary in the sense described in Section II. The initial distribution of the transient driving system in the ensemble is arbitrary. At the initial instant the interaction with the transient driving system is "turned on." We then calculate the mean value of α at subsequent times. It should be emphasized that there is here no selection of a subensemble; i.e., the observer makes no discrimination—he observes whatever happens.

Now let the subscript 1 refer to the combination of driven system and dc driving system and let the subscript 2 refer to the transient driving system. Let the Hamiltonian of the total system be

$$H = H_1 + H_2 + \alpha_1\alpha_2, \tag{20}$$

when H_1 depends only upon the state of the combined system 1 and is identical to H used in previous sections. Likewise, H_2 depends only upon the state of system 2. The interaction term $\alpha_1\alpha_2$ is presumed to produce effects sufficiently small to admit the use of first order time-dependent perturbation theory. The function α_1 is an observable depending only upon the state of system 2 and may be regarded as the transient driving force. The time derivative of α_1, dependent upon state of the combined system 1, is assumed equal to the observable α used previously; i.e.,

$$\dot{\alpha}_1 = \alpha. \tag{21}$$

The operator $\mathcal{L}$ for the total system can be divided into three parts:

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_{12}, \tag{22}$$

where

$$\mathcal{L}_1 = [(\ \ ), H_1],$$
$$\mathcal{L}_2 = [(\ \ ), H_2],$$
$$\mathcal{L}_{12} = [(\ \ ), \alpha_1\alpha_2]. \tag{23}$$

We assume that the statistical ensemble is described initially by a distribution function P. In keeping with earlier remarks we assume that

$$P = P_1^0 P_2, \tag{24}$$

where P_1^0, referring to the combined systems 1, is a distribution in which the driven system may be regarded as stationary although not necessarily in equilibrium. We will consider both cases of equilibrium and nonequilibrium. The distribution function P_2, referring to system 2, is assumed to be completely arbitrary. We introduce for convenience a symbol $\mathcal{S}$ denoting the operation of summing over the states of the total systems. We assume that the class of states summed over is such that $\mathcal{S}$ is factorable in the following way

$$\mathcal{S} = \mathcal{S}_1\mathcal{S}_2. \tag{25}$$

Our problem is now to calculate by first order time-dependent perturbation theory the quantity

$$a(t) = \mathcal{S}P_1^0P_2 \exp\{t(\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_{12})\}\alpha$$
$$\equiv \langle \exp\{t(\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_{12})\}\alpha\rangle. \tag{26}$$

It should be noted that since

$$a^0 = \langle e^{t\mathcal{L}_1}\alpha\rangle = \langle\alpha\rangle = \langle\dot{\alpha}_1\rangle$$
$$= \frac{d}{dt}\,\mathcal{S}_1P_1^0\alpha_1 = 0,$$

the deviations from a^0 are the same as the quantities themselves; e.g., $\Delta a = a$, $\Delta\alpha = \alpha$, etc. The result of first order perturbation theory is

$$a(t) = \int_0^t du\, G(t-u)a_2(u), \tag{27}$$

where

$$G(t) = -\mathcal{S}_1P_1^0(e^{t\mathcal{L}_1}\alpha)[\alpha_1, \log P_1^0]$$
$$\equiv -\langle(e^{t\mathcal{L}_1}\alpha)[\alpha_1, \log P_1^0]\rangle_1, \tag{28}$$

and

$$a_2(u) = \mathcal{S}P_1^0P_2 \exp\{t(\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_{12})\}\alpha_2$$
$$\equiv \langle\exp\{t(\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_{12})\}\alpha_2\rangle. \tag{29}$$

Thus, $G(t-u)$ may be regarded as giving the response to an impulse input at time u. It must be noted that when P_1^0 is a nonequilibrium stationary (in the limited sense, of course) distribution function there is no relation (independent of the particular form of the Hamiltonian) between $G(t)$ derived here and $g(t)$ derived in Section II. *Thus in case of a system whose normal state is stationary but not in equilibrium there is no relation between the response produced by selection and the response produced by perturbation.* Because of this, there is also no relation between noise and the response produced by perturbation.

We will now show that when the combined system 1 is in equilibrium, not merely stationary, $G(t)$ is proportional to $g(t)$. The equilibrium requirement means that

$$P_1^0 = \exp[(A - H_1)/kT], \tag{30}$$

where kT is as before the product of the Boltzmann constant and the absolute temperature, and A is the Helmholtz free energy which plays the role of a normalization constant. With this form of P_1^0 we find

$$[\alpha_1, \log P_1^0] = -[\alpha_1, H_1]/kT = -\dot{\alpha}_1/KT = -\alpha/kT. \tag{31}$$

Substituting this result into (28) and using (10) we obtain

$$G(t) = \langle(e^{t\mathcal{L}_1}\alpha)\alpha\rangle_1/kT$$
$$= g(t)\langle\alpha^2\rangle_1/kT. \tag{32}$$

In deriving (32) we have, of course, used the trivial fact that $\Delta\alpha = \alpha$. It follows further that by choosing c in (12) equal to $(\langle\alpha^2\rangle_1/kT)^{-1}$ we may describe the response $G(t)$ by the same impedance operator $Z(d)/dt$ used in Section III. We obtain

$$Z\left(\frac{d}{dt}\right)[1(t-t')G(t-t')] = \delta(t-t'). \tag{33}$$

By rewriting (27) in the form

$$a(t) = \int_0^\infty du\,[1(t-u)G(t-u)]a_2(u), \tag{34}$$

and operating by $Z(/dt)$, we obtain

$$Z\left(\frac{d}{dt}\right)a(t) = \int_0^\infty du\,\delta(t-u)a_2(u) = a_2(t). \tag{35}$$

Pursuing further the equilibrium case we obtain the Nyquist theorem in either of the following two forms by setting $c = (\langle\alpha^2\rangle_1/KT)^{-1}$

$$C(t-t') = \left\{\left[Z\left(-\frac{d}{dt}\right) + Z\left(\frac{d}{dt}\right)\right]\delta(t-t')\right\}kT \tag{36}$$

$$P(\omega) = \frac{2}{\pi}\,\mathrm{Re}\,Z(i\omega)kT. \tag{37}$$

Design and Performance of Phase-Lock Circuits Capable of Near-Optimum Performance Over a Wide Range of Input Signal and Noise Levels*

R. JAFFE† AND E. RECHTIN†

Introduction

PHASE-LOCK LOOPS provide an efficient method for detection and tracking of narrow-band signals in the presence of wide-band noise. This paper explains how minimum-rms-error loops may be designed if the input-signal level, input-noise level, and a specification for transient performance are given. However, the system performance of such loops departs rapidly from the best obtainable performance if either the signal or the noise levels are different from the design levels, and if no compensating changes are made in the loop. A marked improvement results if the total input power is held constant, regardless of signal or noise levels. It will be demonstrated that a fixed-component loop preceded by a bandpass limiter yields near-optimum performance over a wide range of input signal and noise levels. The following topics will be discussed:

1. An outline of the theoretical design of minimum-rms-error, phase-lock loops when input-signal level, input-noise level, and a specification for transient error are given.
2. The effects of different input levels of signal and noise:
   a. On a system having a fixed-component loop that is optimum only for an original set of levels.
   b. On a system in which loop components maintain optimum performance when the new levels are given.
3. Characteristics of a bandpass limiter.
4. A comparison of the effect of different signal and noise levels:
   a. On a loop using a fixed filter preceded by an automatic-gain-control (AGC) system that holds the signal level constant.
   b. On a fixed-filter loop preceded by a bandpass limiter.
   c. On a variable-filter loop continually adjusted to be optimum.
5. Experimental verification of the fixed-component loop preceded by a bandpass limiter.

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Description of the Loop

The elements of a typical phase-locked loop are shown in Fig. 1. Signal input to such a loop may be assumed to be $\sqrt{2}A \sin [\omega t + \theta_1(t)]$ where A is the rms signal amplitude, ω is the signal-center radian frequency, and $\theta_1(t)$ represents the information content of the signal. It is assumed that the noise input to the loop is narrowbanded about the signal-center frequency and has an essentially flat spectrum over the band; the noise input is represented by $N(t) \sin \omega t$. Output of the voltage-controlled oscillator (VCO) is assumed to have an rms amplitude of C, a center frequency identical to that of the signal, and a phase equal to $\theta_2(t)$.

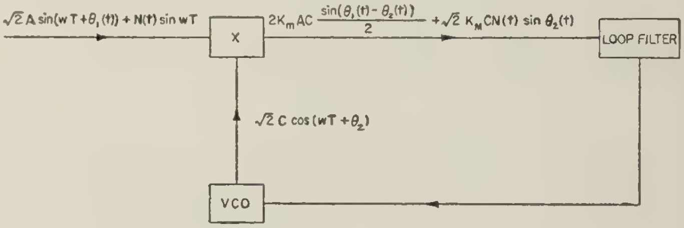


Fig. 1—Typical phase-lock loop.

Briefly, the operating principles of such a loop are as follows: The multiplier beats the signal input and the VCO output together, giving a low-frequency output proportional to the sine of $\theta_1 - \theta_2$. The loop filter accepts only this low-frequency term which is applied as a control voltage to the voltage-controlled oscillator, thereby forcing the VCO output phase θ_2 to be equal to the signal phase θ_1.

It is possible to make a phase-locked loop such that the loop need have a bandwidth only large enough to pass the *difference* between the signal frequency and the VCO estimate of the signal frequency. Since this difference frequency has considerably less variation than the actual signal frequency, the loop does not need nearly as large a bandwidth as would be needed if the loop were merely a tuned circuit placed between the system input and output, which would have to pass all frequencies over which the signal was expected to vary.

Since the bandwidth of the tracking loop is much smaller than that of a comparable nontracking filter, the amount of noise passed on to the output is proportionally smaller, and the loop accordingly picks up a greater resistance to interference.

Linearization of the Loop

To obtain a mathematical description of the loop suitable for subsequent analysis, it is desirable to approximate linearly the nonlinear operation of the actual loop.[1]

The low-frequency multiplier output is

$$2K_M AC \frac{\sin(\theta_1 - \theta_2)}{2} + N(t)\sqrt{2}\, C \sin \theta_2(t),$$

where K_M is the multiplier constant relating the multiplier voltage output both to the amplitude of its two inputs and to their phase difference in radians. Since the filter following the multiplier is low-pass, the high-frequency term in $2\,\omega t$ is disregarded. Under the assumption that the phase error $\theta_1 - \theta_2$ is small and that the input noise is uncorrelated with sine θ_2,

$$\sin(\theta_1 - \theta_2) \cong \theta_1 - \theta_2$$

and

$$N(t) \sin \theta_2 \cong \frac{N(t)}{\sqrt{2}}.$$

The multiplier output may therefore be represented approximately as

$$K_M C[A(\theta_1 - \theta_2) + N(t)].$$

The nonlinear operation of multiplication may now be linearized to subtraction, as shown in Fig. 2, together with associated changes of (1) making the phase input to the loop proportional to $\theta_1 + (N/A)$, (2) inserting an amplifier of gain A after the subtractor so that the voltage feeding the filter will be the same as in the nonlinearized configuration, and (3) representing the VCO as an integrator which relates its phase output to the integral of its input voltage by the VCO constant K_v. The phases around the loop and the transfer functions of the loop elements have been expressed in terms of their Laplace Transforms rather than as functions of time.

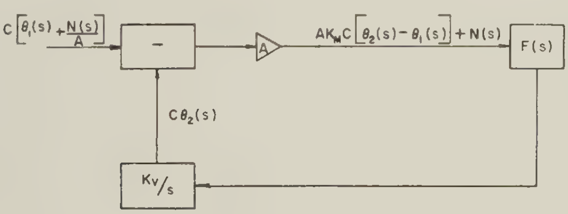


Fig. 2—Partial linearization of phase-lock loop.

Finally, the various gain coefficients may be combined into a single amplifier of gain K, as shown in Fig. 3, where K is evaluated in both radians and degrees. The product AK is defined to be the loop gain.

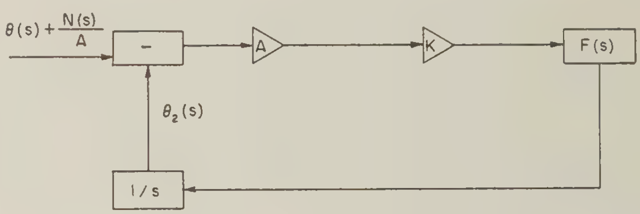
Criteria for Filter Design

The loop filter is important in insuring good loop operation and should be designed so that it performs two distinct functions:

1. It should minimize VCO phase-noise jitter due to noise interference.
2. It should maintain, at a specified amount, transient error in the VCO phase due to specified changes in signal phase.

Transient error is defined as the infinite-time integral of $(\theta_1 - \theta_2)^2$ and is caused solely by signal phase variations. Otherwise expressed,

$$E_T^2 = \int_0^\infty [\theta_1(t) - \theta_2(t)]^2\, dt.$$



$$K = C\,[\text{volts}] \times K_M\,[\text{rad} \times \text{volts}]^{-1} \times K_V \left[\frac{\text{rad/sec}}{\text{volt}}\right]$$

$$= 360 \times C\,[\text{volts}] \times K_M\,[\text{degrees} \times \text{volts}]^{-1} \times K_V \left[\frac{\text{cyc/sec}}{\text{volts}}\right]$$

Fig. 3—Complete linearization of phase-lock loop.

Theory of Filter Design

If the rms phase jitter due to noise interference is denoted by σ_N, and if the transient error, as previously defined, is represented by E_T^2, the design criterion may be stated as follows:

$$\sigma_N^2 + \lambda^2 E_T^2 = \Sigma^2 = \text{minimum},$$

where λ is an undetermined constant, a Lagrangian multiplier used in the standard calculus-of-variations procedure. Theoretically, when the filter form has been computed (Appendix I), the value of λ may be determined from the requirement that the transient error to a specified signal must equal a specified value. Practically, λ is evaluated from the loop-bandwidth parameter B_0, which will be described in a subsequent paragraph.

Filters may be designed for many forms of signal phase inputs. It has been found that filters designed to have zero steady-state error to signal *frequency* changes (zero velocity-error loops) are relatively easy to mechanize and work well for a variety of inputs. This form of filter will be the only one discussed in the body of the paper, although the results of similar analyses concerning loops designed to follow phase steps and frequency ramps are discussed in Appendix II.

Derivation of the optimum filter is given in Appendix I. It is shown that such a filter for zero velocity-error loops has the transform

$$F(s) = \frac{B_0^2 + \sqrt{2}\, B_0 s}{A_0 K s},$$

where

$$B_0^2 \equiv \frac{\lambda(\Delta\omega)}{(N_0/A_0)} \sqrt{2\Delta f},$$

[1] The linearization appears quite legitimate, based on experimental evidence.

and where K represents the loop constant defined in Fig. 3, $\Delta\omega$ is the signal radian-frequency step, N_0 is the expected rms noise input of bandwidth Δf, and A_0 is the expected rms signal strength.

The output phase jitter and transient error of such an optimum loop, when the input noise and signal levels are those expected, are, respectively,

$$\sigma_N^2 = \frac{3}{2\sqrt{2}} B_0\left(\frac{N_0^2}{A_0^2 2\Delta f}\right) \text{rad}^2,$$

and

$$E_T^2 = \frac{(\Delta\omega)^2}{2\sqrt{2}\, B_0^3} \text{rad}^2 \text{ sec}.$$

It may be shown that B_0 is proportional to the expected loop bandwidth; B_0 is approximately equal to two times the loop noise bandwidth and three times the loop 3-db bandwidth. If the loop bandwidth is specified, B_0 may therefore be determined and λ may be evaluated from the definition of B_0.

It is important to note that the parameter B_0 varies with the expected input signal-to-noise ratio and that the value of the optimum filter varies both with B_0 and with the expected signal amplitude A_0.

This filter form is therefore optimum only for a particular input-signal level and a particular input-noise level. If the actual levels differ from their expected values, the filter is no longer optimum.

Performance of Loops Having Levels Other Than Design Levels

It may be shown, by integrating the actual loop transfer function, that the actual output phase jitter and transient error in a fixed-filter loop fed with signal and noise levels different from those for which it was designed are, respectively,

$$\sigma_N^2 = B_0\left(\frac{N^2}{A^2 2\Delta f}\right)\left[\frac{2(A/A_0) + 1}{2\sqrt{2}}\right] \text{rad}^2 \tag{1}$$

and

$$E_T^2 = \frac{(\Delta\omega)^2}{2\sqrt{2}\, B_0^3} \frac{1}{(A/A_0)^2} \text{rad}^2 \text{ sec}, \tag{2}$$

where A is actual signal level, N, actual noise level.

It also may be shown that an optimum variable filter with values continually adjusted with slowly changing signal and noise levels would have the transfer function

$$F(s) = \frac{B_0^2\left[\frac{(A/A_0)}{(N/N_0)}\right] + \sqrt{2}\, B_0\left[\frac{(A/A_0)}{(N/N_0)}\right]^{1/2} s}{KAs}; \tag{3}$$

and noise jitter and transient error, respectively, of

$$\sigma_N^2 = \frac{3B_0}{2\sqrt{2}} \frac{\sqrt{(N_0/A_0)}\,(N/A)^{3/2}}{2\Delta f} \text{rad}^2$$

and

$$E_T^2 = \frac{(\Delta\omega)^2}{2\sqrt{2}\, B_0^3} \frac{(N/A)^{3/2}}{(N_0/A_0)^{3/2}} \text{rad}^2.$$

Thus far, a filter has been derived that is optimum for fixed input levels, and its phase jitter and transient error have been obtained when the input levels matched and mismatched the design levels.

A variable filter has also been derived that was assumed to adjust itself to be optimum for different signal and noise inputs. Expressions for the phase jitter and transient error in this loop have been obtained.

The problem remaining is how to obtain optimum or near-optimum performance over a wide range of input signal and noise levels, yet achieve such performance using a filter than can easily be mechanized—instead of resorting to auxiliary servo loops that continually readjust the filter to keep it optimum.

One possible solution is to use an AGC based on the signal only. The AGC voltage may be obtained if the signal is multiplied by a 90-degree, phase-shifted version of the VCO output; the inputs to the multiplier will then be in phase, and the dc multiplier output will be proportional to the signal level and may be used for the AGC. The AGC method has several disadvantages: (1) It introduces additional components. (2) It introduces additional time constants which may cause two-loop oscillation. (3) It usually results in systems with less dynamic range.

Characteristics of a Bandpass Limiter

An alternate solution, which appeared promising, was to use a limiter preceding the loop. If the noise input to the system increased, the signal strength at the limiter output would decrease because of the limiter property of holding total output power constant. However, the noise bandwidth of the loop is directly dependent on the signal amplitude at the limiter output. Therefore, the increase in input noise, by forcing down the signal amplitude at the limiter output, would reduce the loop bandwidth and require that the phase jitter at the VCO output be a smaller percentage of the input noise. The system would therefore appear to be self-compensating.

The type of limiter to be considered performs perfect snap-action limiting, i.e., its output is assumed to be $+1$ for inputs greater than 0 and -1 for inputs less than 0; such a representation closely approximates an actual limiter fed with input signal and noise levels much greater than its limiting level. The limiter is followed by a filter which restricts the output to the zone centered about the input frequency and excludes the harmonics generated in the limiting action.

Davenport[2] proved that signal-to-noise ratio is essentially preserved in passing through a bandpass limiter. This fact has been experimentally verified at the Jet Propulsion Laboratory. Youla of this laboratory then proved that the total power output of a limiter in a given zone is constant. Mathematically stated:

$$N^2 + A^2 = L^2 = N_0^2 + A_0^2$$

and

$$\frac{N'}{A'} \cong \frac{N}{A},$$

[2] W. B. Davenport, "Signal-to-noise ratios in band-pass limiters," *Jour. of Appl. Phys.*, vol. 24, pp. 720-727; June, 1953.

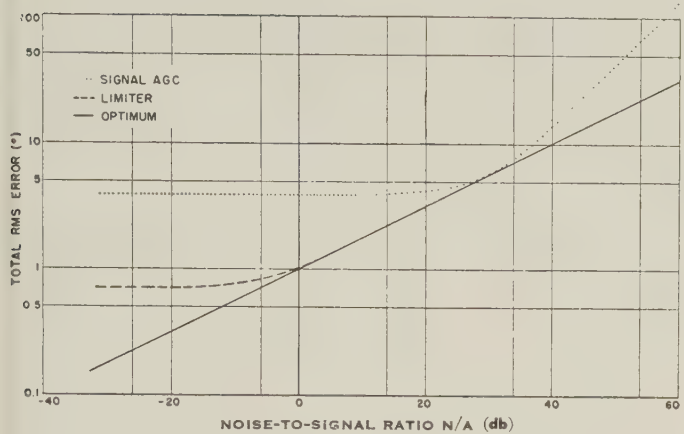


Fig. 4—Total phase error for various types of first-order loops.

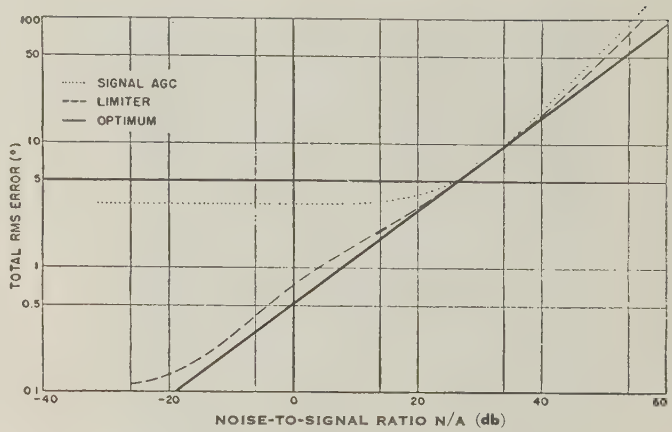


Fig. 5—Total phase error for various types of second-order loops.

where

N' and A' = limiter input noise and signal levels, respectively,

N and A = limiter output noise and signal levels, respectively,

L = total power output in a given zone (constant), and

N_0 , A_0 = output levels of the limiter for which the loop was designed.

These results may be combined to give the ratio both of actual signal level to expected signal level and of actual noise level to expected noise level at the limiter output:

$$\left(\frac{A}{A_0}\right)^2 = \frac{1 + (N_0/A_0)^2}{1 + (N'/A')^2} \qquad \left(\frac{N}{N_0}\right)^2 = \frac{1 + (N_0/A_0)^2}{1 + (A'/N')^2}. \tag{4}$$

This relationship may in turn be applied to the expressions for the phase jitter and transient error [see (1) and (2)] in a fixed loop, giving the phase jitter and transient error in a fixed loop preceded by a limiter.

Comparison of AGC, Limiter, and Variable-Parameter Optimum Loops

It is now desirable to compare the behavior of the various loops discussed. The basis for comparison will be the relative phase jitter, transient error, and total error in the respective loops.

Total rms error is defined as

$$\Sigma = \sqrt{\sigma_N^2 + \lambda^2 \overline{E}_T^2} \text{ rad},$$

and was the quantity minimized by the Wiener calculus-of-variations approach used in designing the filter.

Curves have been plotted for a loop using a fixed filter preceded by an AGC system that holds the signal level constant, for a fixed loop preceded by a limiter and for a variable-filter loop continually readjusted to be optimum. Expressions for loops designed to track input phase steps, frequency steps, and frequency ramps are derived in Appendix II and plotted in Figs. 4, 5, and 6.

Other theoretical curves for a zero velocity-error loop will now be discussed. For these plots the various parameters have been chosen to agree with the experimental

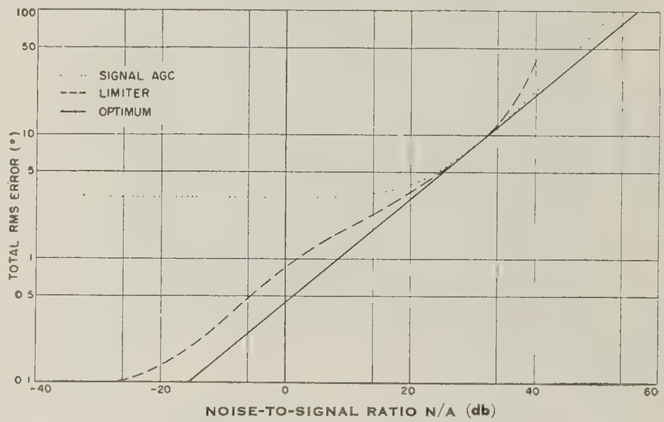


Fig. 6—Total phase error for various types of third-order loops.

and configuration to be described in a later section. Specifically, the input bandwidth of the noise was 900 cycles per second, and the loop noise bandwidth was $12\frac{1}{2}$ cycles per second (i.e., $B_0 = 25$); the loop was designed for an input noise-to-signal ratio of unity.

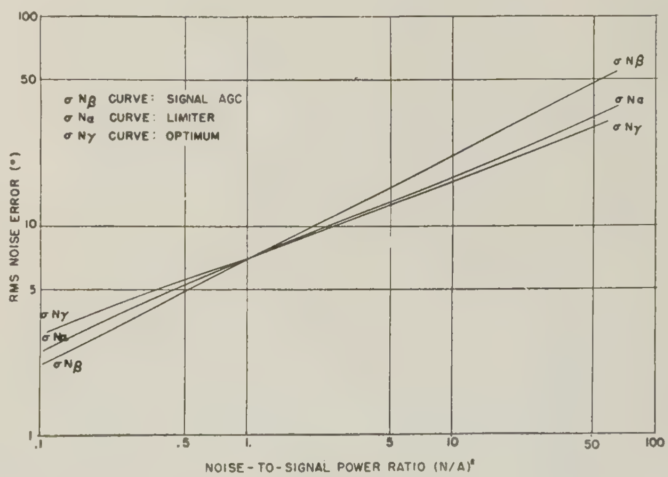


Fig. 7—Comparison of theoretical noise error in various loops.

Fig. 7 shows the rms phase-noise jitter to be expected in the different loops. All the curves coincide at the design point of unity input noise-to-signal ratio but diverge at other input ratios. The phase-noise jitter in a loop employing an optimum variable filter is not necessarily less

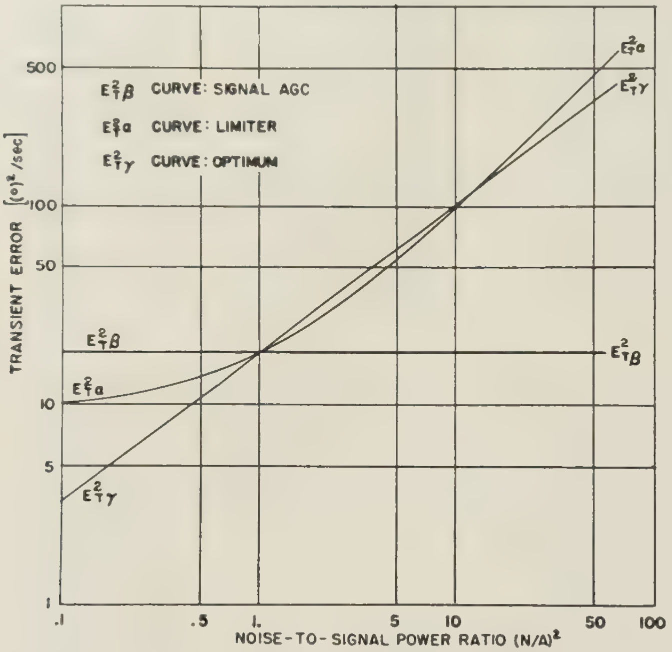


Fig. 8—Comparison of theoretical transient error in various loops.

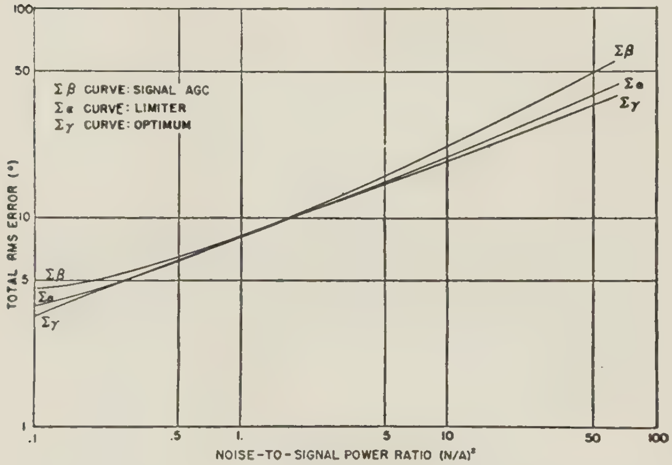


Fig. 9—Comparison of theoretical total error in various loops.

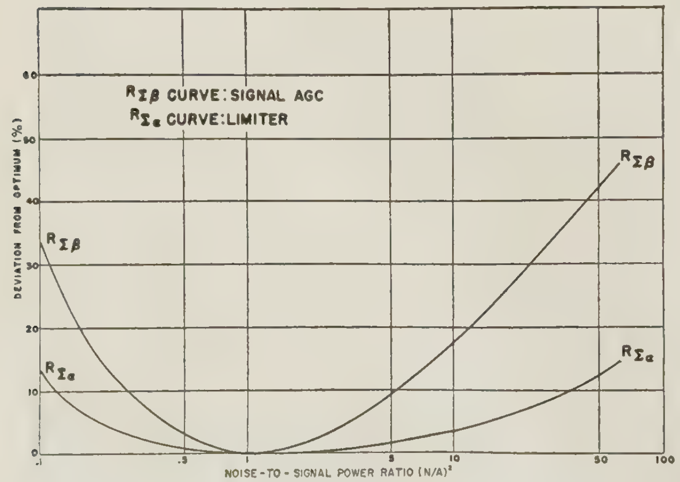


Fig. 10—Total error in signal-AGC and limiter-loop percentage deviation from optimum.

than the jitter in the other loops because the optimum loop is continually minimizing the total error rather than only the phase jitter or the transient error. Optimum performance may require that a slight increase in phase-noise jitter be allowed in order to obtain a considerable decrease in transient error.

Fig. 8 shows the transient error to be expected in the different loops. Since the transient error depends only on the loop gain in a fixed-filter loop tracking a given input frequency step, the AGC loop, which has constant input-signal amplitude, has a fixed loop gain and therefore a fixed amount of transient error.

Fig. 9 shows the total theoretically predicted rms error. However, it is easier to see the significant features of the loops if a plot is made showing the percentage by which the total error in the AGC and limiter loops deviates from the total error in the optimum loop (Fig. 10.) It is seen that the loop preceded by a limiter has a total error not more than 15 per cent greater than optimum, whereas the AGC loop has a total error exceeding that of an optimum loop by 45 per cent.

Experimental Configuration

It has been shown that a limiter loop is theoretically superior to an AGC loop and is nearly as good as a loop continually adjusted to be optimum by auxilliary servos. It remains to be shown that the theoretically derived results for such a limiter loop are experimentally verifiable.

Experimental results were obtained using equipment previously designed.[3] The experimental configuration is shown in Fig. 11. The noise was bandpassed to a 900-cycle

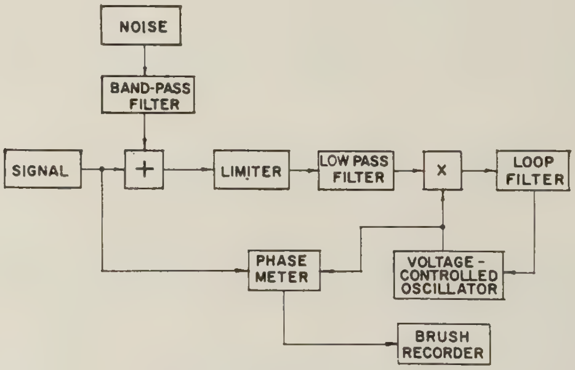
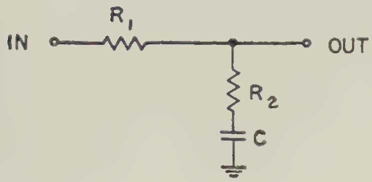


Fig. 11—Experimental configuration for limiter loop.

bandwidth centered at 5 kilocycles by a steep-sided filter and was then added to the 5-kilocycle signal. The sum was fed to a limiter consisting of a four-stage pentode amplifier having back-to-back silicon diodes across the input of each stage. The limiter output was passed thrc h a steep-sided, 7-kilocycle, low-pass filter which remc all but the first zone of the limiter output. The outpu this filter then served as the input to the phase-locked lc

The loop itself consisted of (1) a multiplier using sili diodes in a bridge circuit, (2) a multivibrator-type V(

[3] Equipment used was designed by MacMillan of this laborato



$$\tau_1 \equiv R_1 C \qquad \tau_2 \equiv R_2 C$$

$$\text{DESIRED } F(s) = \frac{B_0^2 + \sqrt{2}\, B_0 S}{K A_0} \qquad \text{ACTUAL } F(s) = \frac{\tau_2 s + 1}{(\tau_1 + \tau_2) s + 1}$$

$$\therefore \tau_1 = \frac{K A_0}{B_0^2} \qquad \cong \frac{\tau_2 s + 1}{\tau_1 s}$$

$$\tau_2 = \frac{\sqrt{2}}{B_0}$$

Fig. 12—Loop filter form.

having a low-pass filter on its output, which passed only the fundamental component of the multivibrator square-wave output, and (3) a loop filter as shown in Fig. 12.

The clean signal was compared with the VCO output in a phase meter, and the recorder output of the phase meter was connected to an oscillograph which was zeroed at the steady no-noise phase difference between the signal and the VCO. Any error in VCO tracking was then plotted directly on the oscillograph.

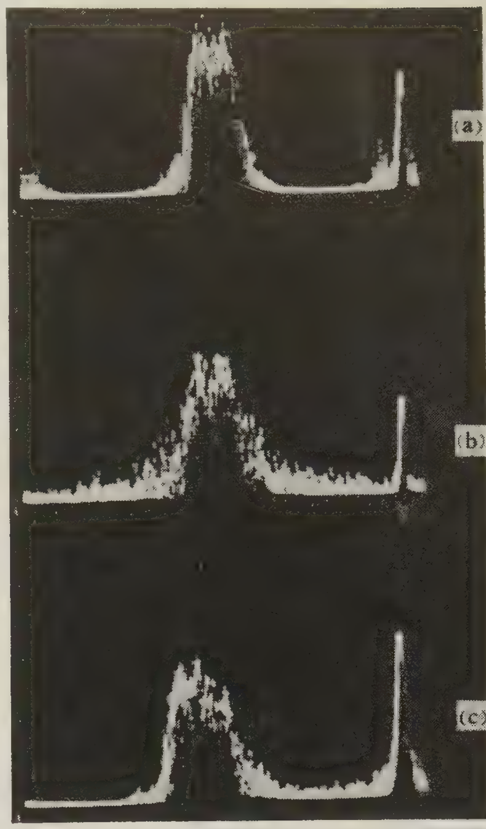


Fig. 13—Noise spectra at various points in limiter loop.

Experimental Results

Before discussing the data obtained from this experiment, it is interesting to look at the spectrum of the noise present in various parts of the loop. Figs. 13(a), 13(b), and 13(c), are, respectively, photographs of the noise spectrum at input to the limiter, at output of the limiter, and at output of the low-pass filter following the limiter.

The spectra are displayed on a logarithmic ordinate, with zero frequency at the right of the photograph. The input noise is centered at 5 kilocycles, with 900 cycles per second bandwidth. The first zone of the limiter output looks quite similar to the input noise except for the presence of some very low-level (considering the logarithmic ordinate) background noise generated in the limiter; also, the limiter input noise has been spread out in a manner similar to that predicted theoretically.[4] The last photograph shows the effect of the 7-kilocycle, low-pass filter in attenuating the limiter background noise past 7 kilocycles, implying that the higher-order noise zones generated in the limiter are cut off.

Experimental data on the noise error were obtained simply by adding a known amount of noise to the signal, and recording the VCO phase jitter on the recorder as shown in Figs. 14(a), 14(b), 14(c), for input noise-to-signal amplitude ratios of 1, 2, and 4, respectively. It is interesting to note how the frequency of the output noise jitter decreases as the input noise-to-signal amplitude ratio increases. This phenomenon implies that the loop bandwidth is decreasing with increasing input noise-to-signal ratio, a result theoretically predicted above.

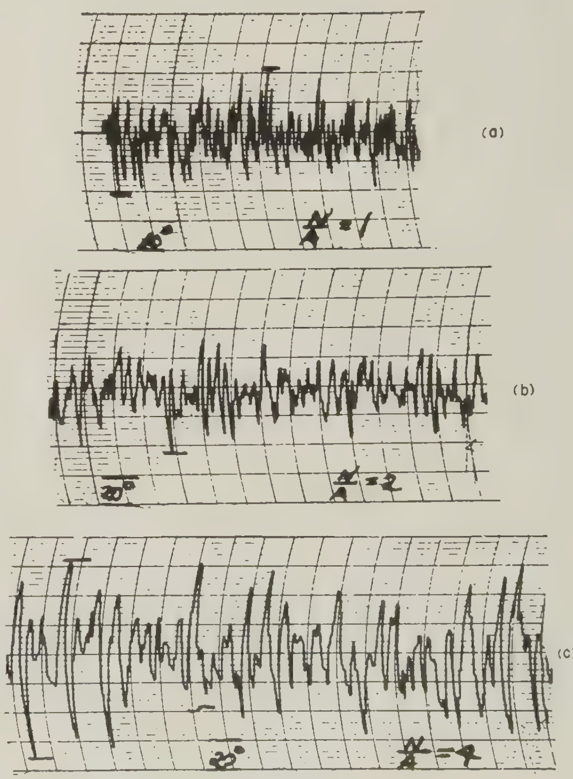


Fig. 14—Experimental phase jitter in limiter loop.

[4] J. L. Lawson and G. E. Uhlenbeck, "Threshold Signals," Radiation Laboratory Series, vol. 24:59, Mass. Inst. Tech., 1950.

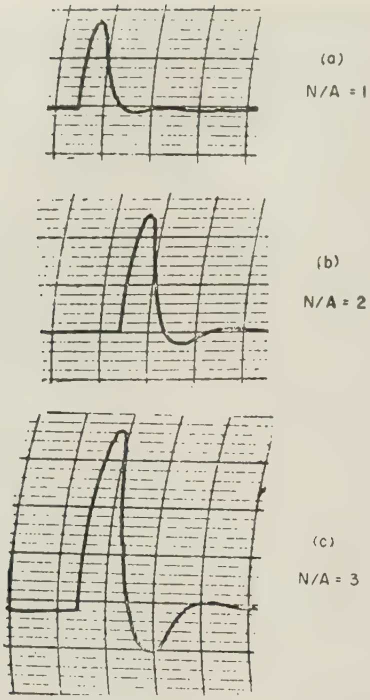


Fig. 15—Experimental transient error in limiter loop.

Experimental data on transient error were more difficult to obtain. In order to determine only the transient performance of the system, it was necessary to disconnect the noise source and reduce the signal output of the limiter in accordance with the limiter relationship already discussed (4). The transient error of the loop was then obtained by introducing a 2.5 cycle per second frequency step into the VCO and recording the loop phase error on the oscillograph. The step could have been introduced into either the signal source or the VCO; the VCO was chosen for convenience.

Figs. 15(a), 15(b), and 15(c) are recordings of the experimental transient error corresponding to input noise-to-signal ratios of 1, 2, and 3, respectively. The ordinate scale is 10 degrees for every heavy line. At the design level of unity signal-to-noise ratio, the Wiener theory requires about 5 per cent overshoot; as was expected, both the initial pip and the overshoot increase as the loop gain and bandwidth decrease.

Finally, the experimental results were combined to plot in Fig. 16, and in Figs. 17 and 18 (opposite) the experimental vs the theoretical values of noise error, transient error, and total error, respectively, in a phase-locked loop preceded by a limiter and employing a fixed filter that satisfied the Wiener optimum criterion for a design level of unity signal-to-noise ratio. The results are commensurate.

Conclusion

It has therefore been theoretically proved that a phase-lock loop preceded by a bandpass limiter approximates, over a wide range of input signal and noise levels, the optimum performance obtainable only with a variable filter that is continually readjusted to be optimum by an auxilliary servo system. It has been shown that, in addition to facility of mechanization, the limiter loop approximates optimum behavior considerably better than does an AGC-controlled loop in which the AGC is based upon the signal. Finally, the theoretical derivations leading to the expressions for error in the limiter loop have been experimentally verified and the effectiveness of a limiter phase-lock loop has been confirmed.

Appendix I

Derivation of Optimum Filter[5]

Let $Y(s)$ be the loop transfer function. By definition

$$\theta_2(s) = Y(s)\theta_1(s) \tag{5}$$

and

$$Y(s) = \frac{AKF(s)}{s + AKF(s)}. \tag{6}$$

The total noise power at the VCO output is

$$\sigma_N^2 = \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} |\,Y(s)\,|^2\,\Phi_N(s)\,ds,$$

where $\Phi_N(s)$ is the spectral density of the input phase noise. Assuming this spectral density to be flat, Figs. 2 and 3 show that

$$\Phi_N(s) = \Phi_N(0) = \frac{N^2}{A^2 2\Delta f}$$

where Δf is the input-noise bandwidth.

The transient error, previously defined as

$$E_T^2 = \int_0^{\infty} [\theta_2(t) - \theta_1(t)]^2\,dt$$

upon application of (5), may be written

$$E_T^2 = \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} |\,Y(s) - 1\,|^2\,|\,\theta_1(s)\,|^2\,ds.$$

Since the optimum filter is to be designed so that σ_N^2 is minimized under the constraint that E_T^2 have a specified value, it is desired that

$$\Sigma^2 = \sigma_N^2 + \lambda^2 E_T^2 = \text{minimum}. \tag{7}$$

The optimum filter is to be physically realizable, which for the purpose of this paper means that the loop transfer function has no poles in the right half of the s plane. Therefore, although the standard variational techniques are to be applied to minimize Σ^2, the integral expression for Σ^2 must be set up in such a manner that it yields a realizable filter.

Expansion of (7) gives

$$\Sigma^2 = \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} ds\; Y(s)Y(-s)\Phi_N(0) + \lambda^2[Y(s) - 1][Y(-s) - 1]\,|\,\theta_1(s)\,|^2. \tag{8}$$

[5] This derivation follows the approach to the Wiener filt heory given in unpublished notes by Rechtin of this laboratory.

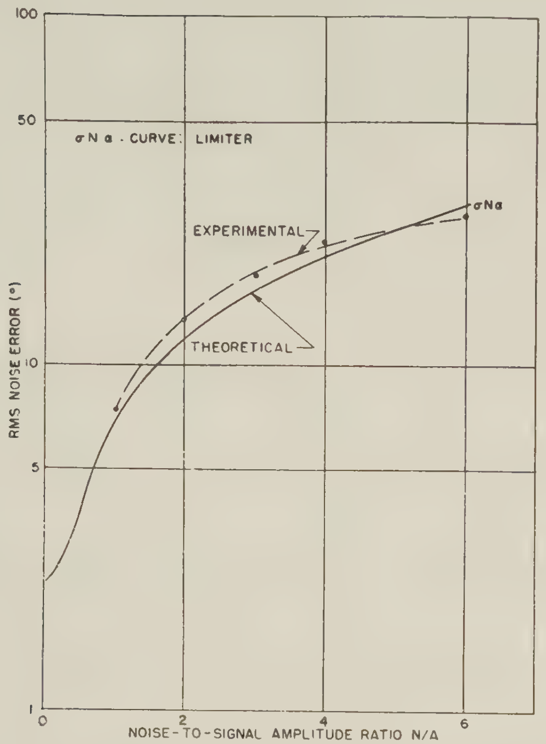


Fig. 16—Experimental vs theoretical noise error in limiter loop.

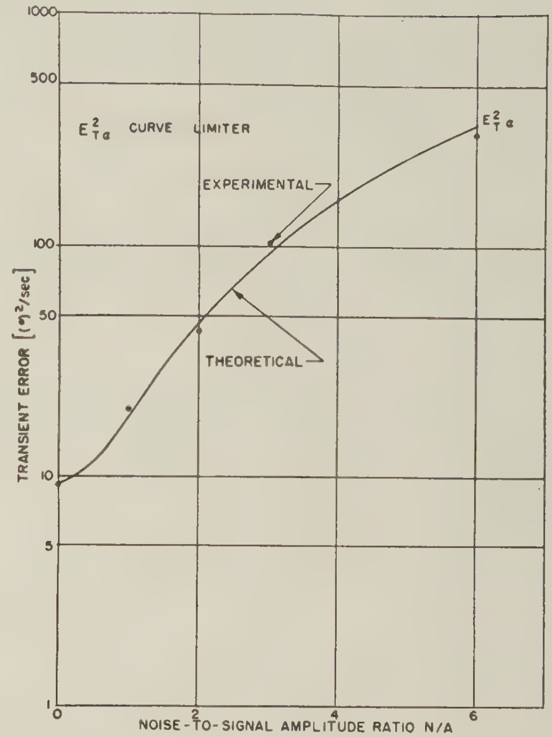


Fig. 17—Experimental vs theoretical transient error for limiter loop.

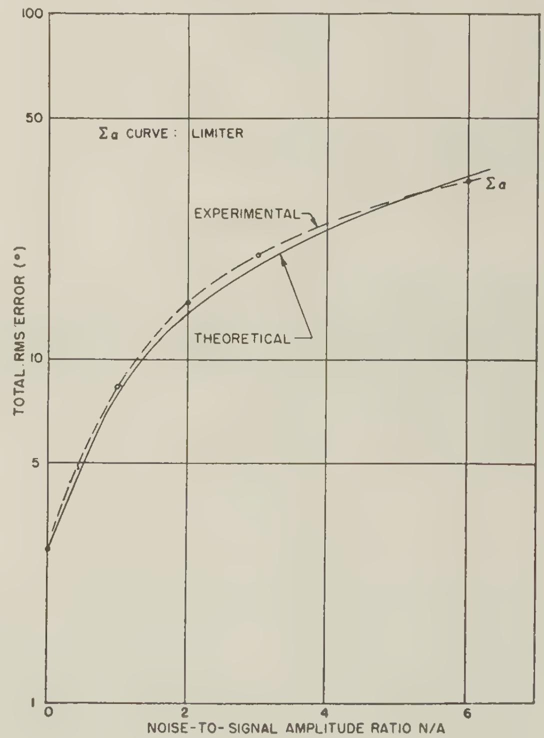


Fig. 18—Experimental vs theoretical total error in limiter loop.

Applying the standard variational procedure to $Y(s)$, let

$$Y^*(s) = Y(s) + \epsilon\eta(s),$$

where $\epsilon\eta(s)$ is the variation to be minimized on the true optimum $Y(s)$, and $Y^*(s)$ is the estimate of $Y(s)$.

When $Y^*(s)$ is split into factors having positive and negative poles and substituted into (8),

$$\begin{aligned}\Sigma^2[Y(s) + \epsilon\eta(s)] &= \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} ds\,[Y(s) + \epsilon\eta(s)][Y(-s) + \epsilon\eta(-s)]\Phi_N(0) \\ &\quad + \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} ds\,\lambda^2[1 - Y(s) - \epsilon\eta(s)] \\ &\quad \cdot [1 - Y(-s) - \epsilon\eta(-s)]\,|\,\theta_1(s)\,|^2.\end{aligned}$$

Setting the variation of Σ^2 to zero at ϵ equals zero completes the standard variational procedure.

$$\begin{aligned}\frac{\partial\Sigma^2}{\partial\epsilon}[Y(s) + \epsilon\eta(s)]\Big|_{\epsilon=0} &= \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} ds\,\eta(s) \\ &\quad \cdot \{(Y(-s))(\lambda^2\,|\,\theta_1(s)\,|^2 + \Phi_N(0)) - \lambda^2\,|\,\theta_1(s)\,|^2\} \\ &\quad + \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} ds\,\eta(-s) \\ &\quad \cdot \{(Y(s))(\lambda^2\,|\,\theta_1(s)^2\,| + \Phi_N(0)) - \lambda^2\,|\,\theta_1(s)\,|^2\} \qquad (9)\end{aligned}$$

In order to keep all terms in (9) split into factors having poles in either, but not both, the right or left half-plane, it is convenient to define

$$\lambda^2|\,\theta_1(s)\,|^2 + \Phi_N(0) \equiv |\,\psi(s)\,|^2 = \psi(s)\psi(-s). \qquad (10)$$

It is now necessary to substitute (10) into (9) in such a way that the latter will be satisfied for realizable $Y(s)$, whereas the remainder of the integral will reduce to zero. This can be effected by noting that

$$\int_{-j\infty}^{+j\infty} Z(s)W(s)\,ds = 0 \qquad (11)$$

if $Z(s)$ and $W(s)$ are algebraic polynomials having poles only in the same half of the s plane.

Substitution of (10) into (9) and keeping together terms having similar poles

$$\int_{-j\infty}^{+j\infty} ds\ \eta(s)\psi(s)[Y(-s)\psi(-s) - \lambda^2 \mid \theta_1(s) \mid^2/\psi(s)]$$
$$+ \eta(-s)\psi(-s)[Y(s)\psi(s) - \lambda^2 \mid \theta_1(s) \mid^2/\psi(-s)] = 0. \quad (12)$$

It is now advisable to split

$$\frac{\mid \theta_1(s) \mid^2}{\psi(s)} = \left[\frac{\theta_1(s)^2}{\psi(s)}\right]_+ + \left[\frac{\theta_1(s)^2}{\psi(s)}\right]_-, \quad (13)$$

where the right-hand side of the equation is the partial fraction expansion of the left-hand side, having denominator terms in $+s$ and $-s$, i.e., having poles in the left and right halves of the s plane, respectively.

Substitution of (13) into (12) gives

$$\int_{-j\infty}^{-j\infty} ds\ \eta(s)\psi(s)\{Y(-s)\psi(-s) - \lambda^2[(\mid \theta_1(s) \mid^2/\psi(s))_+ + (\mid \theta_1(s) \mid^2/\psi(s))_-]\}$$
$$+ \int_{-j\infty}^{+j\infty} ds\ \eta(-s)\psi(-s)\{Y(s)\psi(s) - \lambda^2[(\mid \theta_1(s) \mid^2/\psi(-s))_+ - (\mid \theta_1(s) \mid^2/(-s))_-]\} = 0.$$

Because of (11), the terms having similar poles drop out, leaving

$$\int_{-j\infty}^{+j\infty} ds\ \eta(s)\psi(s)\{Y(-s)\psi(-s) + \lambda^2(\mid \theta_1(s) \mid^2/\psi(s))\}$$
$$+ \int_{j\infty}^{+j\infty} ds\ \eta(-s)\psi(-s) \cdot \{Y(s)\psi(s) - \lambda^2(\mid \theta_1(s) \mid^2/\psi(s))_+\} = 0. \quad (14)$$

These integrals are identical except that one has an integrand in $+s$ and one, in $-s$. Therefore, the integrals are merely the negatives of each other, so that equating one of them to zero satisfies (14).

Setting the second integral to zero yields

$$Y(s) = \frac{\lambda^2}{\psi(s)}\left[\frac{\mid \theta_1(s) \mid^2}{\psi(s)}\right]_+, \quad (15)$$

which is the optimum loop transfer function composed of realizable terms. Note that all terms have poles in the left half-plane, thereby satisfying the requirement of physical realizability.

In this paper, (15) will be solved only for input frequency steps, although the results for other type inputs will be given.

Assume the input to be a frequency step

$$\theta_1(s) = \frac{\Delta\omega}{s^2}$$

where $\Delta\omega$ is the magnitude of the step in radians per second.

Therefore

$$\mid \theta_1(s) \mid^2 = \frac{(\Delta\omega)^2}{s^4}$$

and

$$\mid \psi(s) \mid^2 = \lambda^2 \mid \theta_1(s) \mid^2 + \Phi_N(0) = \frac{\lambda^2(\Delta\omega)^2}{s^4} + \Phi_N(0) = \frac{B_0^4 + s^4}{s^4}\Phi_N(0),$$

where

$$B_0^4 \equiv \frac{\lambda^2\omega^2}{\Phi_N(0)}. \quad (16)$$

Factoring

$$\mid \psi(s) \mid^2 = \psi(s)\psi(-s) = \Phi_N(0)\left[\frac{B_0^2 + \sqrt{2}\,B_0 s + s^2}{s^2}\right]\left[\frac{B_0^2 - \sqrt{2}\,B_0 s + s^2}{s^2}\right]$$

gives

$$\psi(s) = \sqrt{\Phi_N(0)}\left[\frac{B_0^2 + \sqrt{2}\,B_0 s + s^2}{s^2}\right]. \quad (17)$$

Substitution of (17) into the optimum-filter equation gives

$$Y(s) = \frac{\lambda^2 s^2(\Delta\omega)^2}{\Phi_N(0)[B_0^2 + \sqrt{2}\,B_0 s + s^2]}\left[\frac{1}{s^2[B_0^2 - 2B_0 s - s^2]}\right]_+. \quad (18)$$

Separating the last term of (18) into partial fractions,

$$\frac{1}{s^2[B_0^2 + \sqrt{2}\,B_0 s + s^2]} = \frac{1/B_0^2}{s^2} + \frac{\sqrt{2}/B_0^3}{s} + \frac{c}{s - B_0/\sqrt{2}\,(-1+j)} + \frac{d}{s - B_0/\sqrt{2}\,(-1-j)}.$$

The coefficients c and d are of no interest since the terms containing them have poles in the right half-plane and are not to be considered because of the definition of $[\ \ ]_+$ in (13).

The first two terms may be considered as having poles in the left half-plane, since they represent the limiting case of $1/(s - \epsilon)$ where ϵ is a small number representing the reciprocal of the dc gain. Since the pole approaches the origin from the left, in such a limiting process, the first two terms are included in the $[\ \ ]_+$ terms.

Therefore

$$Y(s) = \frac{\lambda^2(\Delta\omega)^2 s^2}{\Phi_N(0)[B_0^2 + \sqrt{2}\,B_0 s + s^2]}\left[\frac{1}{B_0^2 s^2} + \frac{\sqrt{2}}{B_0^3 s}\right],$$

which, with the aid of (16), reduces to

$$Y(s) = \frac{B_0^2 + \sqrt{2}\,B_0 s}{B_0^2 + \sqrt{2}\,B_0 s + s^2}.$$

As a check, it is interesting to note that $Y(s)$ is quasi-distortionless to a step-frequency change, i.e.,

$$\text{error}(s) \equiv \theta_1(s) - \theta_2(s) = \theta_1(s)[1 - Y(s)] = \theta_1(s)\frac{s^2}{B_0^2 + \sqrt{2}\,B_0 s + s^2},$$

so that, if $\theta_1(s) = \Delta\omega/(s^2)$, representing a step-frequency input, the infinite-time error is zero; i.e., the loop is quasi-distortionless to a step-frequency change:

$$\text{error}(s) = \frac{\Delta\omega}{s^2} \frac{s^2}{B_0^2 + \sqrt{2}\,B_0 s + s^2} = \frac{\Delta\omega}{B_0^2 + \sqrt{2}\,B_0 s + s^2}$$

$$\lim_{t\to\infty} \text{error}(t) = \lim_{s\to\infty} s \times \text{error}(s) = 0$$

Since the optimum loop transfer function is

$$Y(s) = \frac{B_0^2 + \sqrt{2}\,B_0 s}{B_0^2 + \sqrt{2}\,B_0 s + s^2}$$

and

$$Y(s) = \frac{AKF(s)}{s + AKF(s)}.$$

It follows that

$$F(s) = \frac{B_0^2 + \sqrt{2}\,B_0 s}{A_0 K s},$$

which was to be proved.

The noise error and transient error in such an optimum loop may be calculated as follows:

$$\sigma_N^2 = \frac{1}{2\pi j} \int_{-j\infty}^{+j\infty} |\, Y(s)\,|^2\, \Phi_N(0)\, ds$$

$$E_T^2 = \frac{1}{2\pi j} \int_{-j\infty}^{+j\infty} |\, \theta_1(s)\,|^2\, [1 - Y(s)]^2\, ds.$$

These integrals may be evaluated by involved contour integration and have been tabulated conveniently;[6] the evaluation of these integrals led to all explicit expressions for σ_N^2 and E_T^2 given in this paper.

Appendix II

Filters for Different Forms of Signal Inputs

Filter forms for fixed-component loops, fixed-component loops preceded by bandpass limiters, and variable-component loops will be given in this appendix, together with their associated transient responses and phase-noise jitters, for two other forms of signal inputs besides those discussed in the text. Derivation of these filters are similar to the derivation in Appendix I and are not repeated. Loops designed to track phase steps will be called first-order loops; loops designed to track frequency ramps will be called third-order loops. Second-order loops designed to track frequency steps are discussed in the text and will not be discussed in this Appendix.

(1) *First-order-loop*—If the signal input to the loop is assumed to be a phase step, i.e.,

$$\theta_1(t) = \Delta\theta \qquad \theta_1(s) = \frac{\Delta\theta}{s},$$

then the filter for the first-order, fixed-component loop has the transform

$$F(s) = \frac{B_1}{KA_0}$$

[6] "Servomechanisms," Radiation Laboratory Series, No. 25, pp. 369–370, Mass. Inst. Tech.; 1947.

and has phase jitter and transient error, respectively, of

$$\sigma_N^2 = B_1\left(\frac{N^2}{2\Delta f A^2}\right)\left(\frac{A}{A_0}\right)$$

and

$$E_T^2 = \frac{(\Delta\theta)^2}{2B_1(A/A_0)}, \tag{19}$$

where B_1 is a quantity similar to B_0 in the previous discussion of the loop for tracking frequency steps, i.e.,

$$B_1 \equiv \frac{\lambda(\Delta\theta)}{N_0/A_0}\sqrt{2\Delta f}.$$

Substitution of (4) into (19) gives a total error in a first-order loop preceded by a bandpass limiter of

$$\Sigma^2 \equiv \sigma_N^2 + \lambda^2 E_T^2 = \left(\frac{B_1}{2}\right)\left(\frac{N^2}{A^2 2\Delta f}\right)\left(\frac{A}{A_0}\right) + \left(\frac{B_1}{2}\right)\left(\frac{N_0^2}{A_0^2 2\Delta f}\right)\left(\frac{A_0}{A}\right)$$

$$= \frac{B_1}{2}\left(\frac{N^2}{A^2 2\Delta f}\right)\left[\sqrt{\frac{1 + (N_0/A_0)^2}{1 + (N/A)^2}} + \frac{N_0^2}{A_0^2}\sqrt{\frac{1 + (N/A)^2}{1 + (N_0/A_0)^2}}\right].$$

It may be shown, using the same techniques employed in Appendix III for second-order loops, that the optimum variable-component filter for the first-order loop has the transform

$$F(s) = \frac{B_1}{KA_0}\frac{N_0}{N},$$

and that the total error in such a loop is

$$\Sigma^2 \equiv \sigma_N^2 + \lambda^2 E_T^2 = \left(\frac{B_1}{2}\right)\left(\frac{A/A_0}{N/N_0}\right)\left(\frac{N^2/A^2}{2\Delta f}\right) + \left[\frac{\lambda^2(\Delta\theta)^2}{2B_1}\right]\frac{N/N_0}{A/A_0}$$

$$= B_1\left(\frac{N_0}{A_0}\right)\left(\frac{N/A}{2\Delta f}\right).$$

Plots for the total error in first-order loops with (1) fixed components preceded by a signal-based AGC, (2) fixed components preceded by a bandpass limiter and (3) variable components are given in Fig. 4. Design parameters are

$$\frac{N_0}{A_0} = 1$$

$$B_0 = 10$$

$$\Delta f = 1 \text{ megacycle per second.}$$

The input-signal change was assumed to be a 1-radian phase step.

For comparison purposes, the total error in each of the three types of second-order loops is given in Fig. 5. The input-signal change was assumed to be a 6 radians per second ($\sim$ 1 cycle per second) frequency step.

(2) *Third-order-loop*—If the signal input to the loop is assumed to be a frequency ramp, ie.,

$$\theta_1(t) = \frac{(\Delta\alpha)t^2}{2} \qquad \theta_1(s) = \frac{\Delta\alpha}{s^3},$$

then the third-order, fixed-component loop filter has the transform

$$F(s) = \frac{B_2^3 + 2B_2^2 s + 2B_2 s^2}{KA_0 s^2}$$

and has phase jitter and transient error, respectively, of

$$\sigma_N^2 = \left(\frac{N^2}{A^2}\right)\left(\frac{B_2}{2\Delta f}\right)\left[\frac{4(A/A_0) + 1}{4(A/A_0) - 1}\right]$$

and

$$E_T^2 = \frac{(\Delta\alpha)^2}{(A/A_0)[4(A/A_0) - 1]B_2^3}, \tag{20}$$

where

$$B_2^3 \equiv \left[\frac{\lambda(\Delta\alpha)}{N_0/A_0}\right]\sqrt{2\Delta f}.$$

Substitution of (4) into (20) gives a total error in a third-order loop preceded by a band-pass limiter of

$$\Sigma^2 \equiv \sigma_N^2 + \lambda^2 E_T^2 = \left(\frac{B_2}{2}\right)\left[\frac{N^2}{A^2 2\Delta f}\right]\left[\frac{4(A/A_0) + 1}{4(A/A_0) - 1}\right] + \left(\frac{B_2}{2}\right)\left[\frac{N_0^2}{A_0^2 2\Delta f}\right]\frac{(A/A_0)}{[4(A/A_0) - 1]}$$

$$= B_2\left[\frac{N^2}{A^2 2\Delta f}\right]\left\{\left[\frac{4\sqrt{\dfrac{1 + (N_0/A_0)^2}{1 + (N/A)^2}} + 1}{4\sqrt{\dfrac{1 + (N_0/A_0)^2}{1 + (N/A)^2}} - 1}\right]\left(\frac{A}{A_0}\right) + \frac{(A/A_0)^2}{(N/N_0)^2}\left[\frac{1}{\sqrt{\dfrac{1 + (N_0/A_0)^2}{1 + (N/A)^2}}\left[4\sqrt{\dfrac{1 + (N_0/A_0)^2}{1 + (N/A)^2}} - 1\right]}\right]\right\}.$$

It should be noted that the total error becomes infinite when the noise has reduced the signal level to one-fourth of the design level.

It may be shown that the optimum third-order, variable-component loop has the transform

$$F(s) = \frac{B_2^3\left[\dfrac{(A/A_0)}{(N/N_0)}\right] + 2B_2^2\left[\dfrac{(A/A_0)}{(N/N_0)}\right]^{2/3} s + 2B_2\left[\dfrac{(A/A_0)}{(N/N_0)}\right]^{1/3} s^2}{As^2},$$

and that the total error in such a loop is

$$\Sigma^2 = \sigma_N^2 + \lambda^2 E_T^2 = \left(\frac{5}{3}\right)\left(\frac{N}{A}\right)^{5/3}\left(\frac{N_0}{A_0}\right)^{1/3}\left(\frac{B_2}{2\Delta f}\right) + \frac{(\Delta\alpha)^2}{3B_2^5\left[\dfrac{A/A_0}{N/N_0}\right]^{5/3}}$$

$$= 2B_0\left(\frac{N^2}{A^2 2\Delta f}\right)\left[\frac{(N_0/N)}{(A_0/A)}\right]^{1/3}.$$

Plots for the total error in the three types of third-order loops are given in Fig. 6. The input-signal change was assumed to be a 1 radian per second^2 frequency ramp.

Appendix III

Optimum-Filter Design

The optimum filter is specified by

$$F(s) = \frac{B^2 + \sqrt{2}\,Bs}{KAs},$$

where

$$B^2 \equiv \frac{\lambda(\Delta\omega)}{(N/A)}\sqrt{2\Delta f}.$$

B varies in accordance with the signal- and noise-input levels and may be expressed in terms of B_0, the fixed-filter bandwidth parameter, as follows:

$$B^2 = B_0^2\left[\frac{(A/A_0)}{(N/N_0)}\right].$$

Therefore

$$F(s) = \frac{B_0^2\left[\dfrac{(A/A_0)}{(N/N_0)}\right] + \sqrt{2}\,B_0\left[\dfrac{(A/A_0)}{(N/N_0)}\right]s}{KAs}, \tag{21}$$

giving transfer function for optimum variable filter (3).

All results described in this paper as pertaining to second-order, variable-filter loops were obtained by substituting (21) into the expression of $Y(s)$ in (6) and evaluating that which was required.

INFORMATION FOR AUTHORS

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Papers should be written concisely, keeping to a minimum all introductory and historical material. It is seldom necessary to reproduce in their entirety previously published derivations, where a statement of results, with adequate references, will suffice.

To expedite reviewing procedures, it is requested that authors submit the original and two legible copies of all written and illustrative material. The manuscript should be double-spaced, and the illustrations drawn in india ink on drawing paper or drafting cloth. Each paper should include a carefully written abstract of not more than 200 words. Upon acceptance, papers should be prepared for publication in a manner similar to those intended for the PROCEEDINGS OF THE IRE. Further instructions may be obtained from the Publications Chairman. Material not accepted for publication will be returned.

IRE TRANSACTIONS ON INFORMATION THEORY is published three times a year, in March, September, and December. A minimum of one month must be allowed for review and correction of all accepted manuscripts. A period of approximately two months additional is required for the mechanical phases of publication and printing. Therefore, all manuscripts must be submitted three months prior to the respective publication dates. There will be no June issue, since the IRE CONVENTION RECORD is published at that time, and a bound collection of Information Theory papers previously delivered at the 1955 Convention will be mailed gratis to all PGIT members.

All technical manuscripts and editorial correspondence should be addressed to Laurin G. Fischer, Federal Telecommunications Labs., 492 River Road, Nutley, N. J. Local Chapter activities and announcements, as well as other nontechnical news items, should be addressed to Nathan Marchand, Marchand Electronic Labs., 255 Mill Street, Byram, Conn.